On Automorphisms of Distance-Regular Graph with Intersection Array {99,84,1;1,12,99}

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We consider undirected graphs without loops or multiple edges. Given a vertex a in a graph Γ, let denote the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all its vertices that are a distance of i away from a. Let $[a] = \Gamma_1(a)$ and $a^\perp = \{a\} \cup [a]$.

If u and w are vertices separated by a distance of i in Γ, then $b_i(u, w)$ ($c_i(u, w)$) denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called a distance-regular graph with an intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of vertices u and w separated by a distance of i in Γ for any $i = 0, ..., d$. Let $a_i = k - b_i - c_i$. Note that, for a distance-regular graph, b_0 is the degree of the graph and $c_1 = 1$. Given a subset X of automorphisms of Γ, let $\mathrm{Fix}(X)$ denote the set of all vertices of Γ that are fixed under any automorphism from X. Let $p^l_{ij}(x, y)$ denote the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x and y separated by a distance of l in Γ. In a distanceregular graph, the numbers $p^l_{ij}(x, y)$ are independent of the choice of x and y; they are denoted by p^l_{ij} and are known as the intersection numbers of Γ.

Let $\alpha_i(g)$ denote the number of points $u \in \Gamma$ such that $d(u, u^g) = i$ for $g \in \mathrm{Aut}(\Gamma)$.

Intersection arrays distance-regular graphs, in which neighborhoods of vertices are strongly regular with parameters (99,14,1,2) were found in [1]: $\{99, 84, 1; 1, 12, 99\}$, $\{99, 84, 1; 1, 14, 99\}$, $\{99, 84, 30; 1, 6, 54\}$.

These abstracts are considered possible orders and subgraphs of fixed points hypothetical distance-regular graph with intersection array $\{99, 84, 1; 1, 12, 99\}$.

Theorem. Let Γ be a distance-regular graph with the intersection array $\{99, 84, 1; 1, 12, 99\}$, $G = \mathrm{Aut}(\Gamma)$, g be an element of prime order p in G, and $\Omega = \mathrm{Fix}(g)$ contains at s vertices in the t antipodal classes. Then $\pi(G) \subseteq \{2, 3, 5, 7, 11\}$ and one of the following assertions holds:

(1) $s = 0$ and either
  (*i*) $p = 5$, $\alpha_1(g) = 100l$, $\alpha_2(g) = 800 - 100l$, $\alpha_3(g) = 0$, where $0 \leq l \leq 8$ or
  (*ii*) $p = 2$, $\alpha_3(g) = 16l$, $\alpha_1(g) = 16l - 40m$ for some $0 \leq l \leq 50$;
(2) $p = 11$, $t = 1$ and $\alpha_1(g) = 220l - 44$;
(3) $p = 7$, Ω is a t-clique and either
  (*i*) $t = 2$, $\alpha_3(g) = 14$, $\alpha_1(g) = 140l + 98$ or
  (*ii*) $t = 9$, $\alpha_3(g) = 63$, $\alpha_1(g) = 140l - 49$;
(4) $p = 5$, $s = 3$, $\alpha_0(g) = 3t$ и $t = 15, 20, ..., 35$;
(5) $p = 3$ and either
  (*i*) $s = 2$ and $t \in \{1, 4, ..., 25\}$ or
  (*ii*) $s = 5$ and $t = 1, 4, 7, ..., 22$ or
  (*iii*) $s = 8$ and $t = 1, 4, 7, 10, 13$;
(6) $p = 2$, t is even and either
  (*i*) $s = 2$ and $t \leq 28$ or
  (*ii*) $s = 4$ and $t \leq 28$ or
  (*iii*) $s = 6$ and $t \leq 18$ or
  (*iv*) $s = 8$ и $t \leq 12$.

Corollary. Let Γ be a distance-regular graph with the intersection array $\{99, 84, 1; 1, 12, 99\}$ and $G = \mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices graph Γ. Then G is a $\{2, 3, 5\}$-group.

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References

[1] Makhnev A.A., On graphs whose local subgraphs are strongly regular with parameters (99,14,1,2). *Doklady Mathematics* **88**:3 (2013) 737-740.