Automorphisms of strongly regular graph with parameters (1197,156,15,21)

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We consider nondirected graphs without loops amd multiple edges. For vertex a of a graph Γ the subgraph $\Omega_i(a) = \{b \mid d(a,b) = i\}$ is called i-neighboorhod of a in Γ. We set $[a] = \Gamma_1(a)$, $a^\perp = \{a\} \cup [a]$.

Degree of an vertex a of Γ is the number of vertices in $[a]$. Graph Γ is called regular of degree k, if the degree of any vertex is equal k. The graph Γ is called amply regular with parameters (v,k,λ,μ) if Γ is regular of degree k on v vertices, and $|[u] \cap [w]|$ is equal λ, if u adjacent to w, is equal μ, if $d(u,w) = 2$. Amply regular graph of diameter 2 is called strongly regular.

A partial geometry $pG_\alpha(s,t)$ is a geometry of points and lines such that every line has exactly $s+1$ points, every point is on $t+1$ lines (with $s > 0,\ t > 0$) and for any antiflag (P,y) there are exactly α lines z_i containing P and intersecting y. In the case $\alpha = 1$ we have generalized quadrangle $GQ(s,t)$. The incidence system $(X,\mathcal{B})$ with a point-set X and block-set $\mathcal{B}$ is called t-(V,K,Λ) *design*, if $|X| = V$, each block contains exactly K points and any t points belong to exactly Λ blocks. Every 2-design is (V,B,R,K,Λ) design, where $B = |\mathcal{B}|$, each point belong to exactly R blocks, and we have equalities $VR = BK$, $(V-1)\Lambda = R(K-1)$. Design is symmetric, if $B = V$. Design is called quasi-symmetric, if for every two blocks $B, C \in \mathcal{B}$ we have $|B \cap C| \in \{x,y\}$. Numbers x,y are called intersection numbers of quasi-symmetric design, and it is suggested that $x < y$.

Block-graph of quasi-symmetric design $(X,\mathcal{B})$ have as a vertex set $\mathcal{B}$ and two blocks $B, C \in \mathcal{B}$ are adjacent, if $|B \cap C| = y$.

Proposition 1 ([1], theorem 5.3). *Block-graph of quasi-symmetric* (V,B,R,K,Λ) *design is strongly regular with spectrum* $((R-1)K - xB + x)/(y-x)^1$, $(R-K-\Lambda+x)/(y-x)^{V-1}$, $-(K-x)/(y-x)^{B-V}$.

Derived design for t-(V,K,Λ) design $\mathcal{D} = (X,\mathcal{B})$ at $x \in X$ is design $\mathcal{D}_x$ with the point-set $X_x = X - \{x\}$ and block-set $\mathcal{B}_x = \{B - \{x\} \mid x \in B \in \mathcal{B}\}$. Designe $\mathcal{E}$ is called an extension of $\mathcal{D}$, if derived design of $\mathcal{E}$ at each point is isomorphic to $\mathcal{D}$. Residual design of $\mathcal{D}$ at a block B is the design $\mathcal{D}^B$ with the point-set $X^B = X - B$ and block-set $\mathcal{B}^B = \{C \in \mathcal{B}\} \mid |B \cap C| = 0\}$.

It is known that projective plane is extenable if and only if its order is 2 or 4. P. Cameron ([1], theorem 1.35) classified extensions of symmetric 2-designs.

Proposition 2. *Let* 3-(V,K,Λ) *design* $\mathcal{E} = (X,\mathcal{B})$ *is an extension of symmetric 2-design. Then one of the following holds:*

(1) $\mathcal{E}$ *is the Hadamard* 3-$(4\Lambda+4, 2\Lambda+2, \Lambda)$ *design;*

(2) $V = (\Lambda+1)(\Lambda^2+5\Lambda+5)$ *and* $K = (\Lambda+1)(\Lambda+2)$*;*

(3) $V = 496$*,* $K = 40$ *and* $\Lambda = 3$.

In the case (3) we have $R = V - 1 = 495$, $B = VR/K = 496 \cdot 495/40 = 6138$ and the complement to block-graph has parameters (6138,1197,156,252) and spectrum $1197^1, 9^{5642}, -105^{495}$. Hence maximal order of coclique is at most $vm/(k+m) = 6138 \cdot 105/1302 = 495$. In particular, the Hoffman bound is equal to Cvetkovich bound. The complement graph to block-graph of 3-(496,40,3) design is called Cameron monster. In [2] it is proved

Proposition 3. *For Cameron monster* Γ *the following hold:*

(1) *neighborhood of every vertex of* Γ *is strongly regular graph with parameters* $(1197, 156, 15, 21)$ *and spectrum* $156^1, 9^{741}, -15^{455}$*, and the order of coclique in this graph is at most* 105*;*

(2) *the set of blocks* C_x *containing apoint* x *of designe* $\mathcal{E}$ *is* 495*-coclique of* Γ*, for which the equality holds in Hoffman bound and Cvetkovich bound;*

(3) *subgraph* $\Gamma - C_x$ *is strongly regular graph with parameters* $(5643, 1092, 141, 228)$ *and spectrum* $1092^1, 9^{5148}, -96^{494}$*;*

(4) *for distinct points* x,y *of design* $\mathcal{E}$ *we have* $|C_x \cap C_y| = 39$*, and for coclique* $C_x - C_y$ *of graph* $\Gamma - C_y$ *the equality holds in Hoffman bound.*

In this paper automorphisms of strongly regular graph with parameters $(1197, 156, 15, 21)$ are founded.

Theorem. *Let* Γ *be a strongly regular graph with parameters* $(1197, 156, 15, 21)$*,* $G = \mathrm{Aut}(\Gamma)$*,* g *an element of prime order* p *of* G *and* $\Omega = \mathrm{Fix}(g)$*. Then* $|\Omega| \leq 171$*,* $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13, 19\}$ *and one of the following holds:*

(1) Ω *is empty graph, eitrher* $p = 3$ *and* $\alpha_1(g) = 72l$*, or* $p = 7$ *and* $\alpha_1(g) = 168l - 21$*, or* $p = 19$ *and* $\alpha_1(g) = 456l + 171$*;*

(2) Ω *is* n*-clique, end either*

(i) $p = 13$*,* $n = 1$ *and* $\alpha_1(g) = 312l + 156$*, or*

(ii) $p = 2$*,* $n = 9$ *and* $\alpha_1(g) = 48l + 12$ *or* $n = 11$ *and* $\alpha_1(g) = 32l - 12$*, or*

(iii) $p = 5$*,* $n = 2$ *and* $\alpha_1(g) = 120l + 45$ *or* $n = 7$ *and* $\alpha_1(g) = 120l - 30$*;*

(3) Ω *is* $3t + 1$*-coclique,* $p = 3$ *and* $\alpha_1(g) = 72l + 12 - 45t$*;*

(4) Ω *contains geodesic* 2*-way and* $p \leq 13$.

Corollary. *Strongly regular graph with parameters* $(1197, 156, 15, 21)$ *is not vertex-symmetric.*

This work was supported by the grant of Russian Science Foundation, project no. 15-11-10025

References

[1] P. Cameron, J. Van Lint, Designs, Graphs, Codes and their Links. *London Math. Soc. Student Texts, N 22, Cambridge: Cambr. Univ. Press* 1981, 240 p.

[2] A. Makhnev, Extensions of symmetric 2-designs. *Maltsev chteniya, Abstracts of Intern. Conf. Novosibirsk* 2015, 111.