Faithful representations of the strong endomorphism monoid of graphs and n-uniform hypergraphs

Eugenja Bondar
Luhansk Taras Shevchenko National University, Luhansk
Ural Federal University, Ekaterinburg, Russia

U. Knauer and M. Nieporte [1] proved that the monoid of strong endomorphisms of any finite undirected graph without multiple edges is isomorphic to the wreath product of a monoid with a certain small category. It was shown in [1] also that the representation fails in infinite case. In [2] we have defined a certain class of infinite undirected graphs and a certain class of infinite n-uniform hypergraphs and found faithful representations of the strong endomorphism monoid of graphs and hypergraphs from these classes. Here we generalize results of [2].

Let $G = (V, E)$ be an infinite undirected graph without multiple edges. Recall that a transformation $\varphi : V \to V$ is called a *strong endomorphism* of G if $\{x, y\} \in E \Leftrightarrow \{x\varphi, y\varphi\} \in E$ for all $x, y \in V$. The set of all strong endomorphisms of a graph G forms a monoid under composition and is denoted by $\mathrm{SEnd}\, G$. By $N(x)$ we denote the neighborhood of a vertex $x \in V$, that is, the set $\{y \in V \mid \{x, y\} \in E\}$. Let ν be the equivalence on V defined by $x \ \nu \ y \Leftrightarrow N(x) = N(y)$ for $x, y \in V$. The ν-class that contains x is denoted by x_ν. The graph G/ν with the vertex set V/ν and the edge set $\{\{a_\nu, b_\nu\} \mid \{a, b\} \in E\}$ is called the canonical strong quotient graph of the graph G.

A hypergraph is a pair $(V, \mathcal{E})$, where V is a nonempty set of elements called vertices and $\mathcal{E}$ is a family of nonempty subsets of V called edges. A hypergraph H is called an n-uniform hypergraph if it has no multiple edges and each edge consists of exactly n vertexes. By C_n we denote the class of all n-uniform hypergraphs. A transformation $\alpha : V \to V$ of a hypergraph $H \in C_n$ is called a *strong endomorphism* of the hypergraph if $A \in \mathcal{E} \Leftrightarrow A\alpha \in \mathcal{E}$ for all $A \subseteq V$, $|A| = n$. The set of all strong endomorphisms of a hypergraph H forms a monoid under composition and is denoted by $\mathrm{SEnd}\, H$.

Let $H \in C_n$ and x be a vertex of H. A neighborhood of x is defined by the formula $\mathcal{N}(x) = \{A \subseteq V : |A| = n - 1, \ A \cup \{x\} \in \mathcal{E}\}$. By $\rho(x)$ we denote the number of edges that contain x. For an arbitrary hypergraph $H \in C_n$ we define the equivalence relation ν on its vertex set by the rule:

$$x \ \nu \ y \Leftrightarrow \mathcal{N}(x) = \mathcal{N}(y) \text{ if } \ n \geq 2, \text{ and } x \ \nu \ y \Leftrightarrow \rho(x) = \rho(y) \text{ if } \ n \in \{0, 1\}.$$

Let H/ν be the hypergraph whose vertex set equals V/ν, and edge set consists of all $A_\nu = \{a_\nu \mid a \in A\}$, $A \subseteq V$ such that there exists a transversal T of the family A_ν with $T \in \mathcal{E}(H)$. The hypergraph H/ν is called the canonical strong quotient hypergraph of the hypergraph H.

Let $\mathrm{SEnd}_\nu\, G \subseteq \mathrm{SEnd}\, G$ ($\mathrm{SEnd}_\nu\, H \subseteq \mathrm{SEnd}\, H$) be the set of the strong endomorphisms of G (respectively H) that preserve the relation ν.

Theorem. *For an arbitrary infinite undirected graph without multiple edges G (infinite n-uniform hypergraph H) the set $\mathrm{SEnd}_\nu\, G$ ($\mathrm{SEnd}_\nu\, H$) constitutes a submonoid of $\mathrm{SEnd}\, G$ ($\mathrm{SEnd}\, H$), which is isomorphic to a wreath product of all strong injective endomorphism monoid of G/ν (H/ν) with a certain small category.*

The aforementioned results of [2] are immediate consequences of our theorem. Moreover, the theorem is true for arbitrary graphs without multiple edges and n-uniform hypergraphs.

References

[1] Knauer U., Nieporte M., Endomorphisms of graphs I. The monoid of strong endomorphisms// *Arch. Math.* **52** (1989) 607-614.

[2] Bondar E.A., Zhuchok Yu.V., Semigroups of the strong endomorphisms of infinite graphs and hypergraphs // *Ukr.Math.J.* **65(6)** (2013) 743-754.