Some combinatorial problems in symmetric groups

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One of the actual problems in the group theory is a representation of an element of the group by the word of generators. For example, the study of infinite groups saturated by a set of finite groups is this one. Also these tasks are arised in the many practical problems. For example, in the design of the topology of a multiprocessor computing system (MCS). In this case the model of MCS will be presented as the Cayley graph in which the the processors are the vertices of the graph and the edges correspond to physical connections between processors.

Let $S_n =$ be the symmetric group of degree n and $x = (1, 2)$, $y = (1, 2, \dots, n)$ be generators of S_n. Let $x < y$ and elements of S_n written by words of generators be lenlex ordered. $\pi_i(n)$ denote elements of S_n which have the maximal length in this ordering. Our hypothesis for $n \geq 6$ is following.

1. For even n there is the only one permutation $\pi(n)$:

$$\pi(n) = (1, 3, n, 2) \prod_{i=1}^{\frac{n-4}{2}} (a_i, b_i), \quad a_i = 3 + i, \quad b_i = n - i.$$

2. For odd n there are the only two permutations $\pi_1(n)$ and $\pi_2(n)$:

$$\pi_1(n) = (1, 2)(a_1, b_1, a_2, b_2, \dots, \frac{n+3}{2}), \quad a_i = 2 + i, \quad b_i = n + 1 - i, \quad i \leq \frac{n-3}{2};$$

$$\pi_2(n) = (1, 2)(3, n, \frac{n+3}{2}, a_1, b_1, a_2, b_2, \dots), \quad a_i = \frac{n+3}{2} - i, \quad b_i = \frac{n+3}{2} + i, \quad i \leq \frac{n-5}{2}.$$

The following table shows examples of $\pi_i(n)$ for $6 \leq n \leq 12$ which are obtained by computer computations.

Group	Permutations $\pi_i(n)$	Product of cycles $\pi_i(n)$
S_6	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 5 & 4 & 2 \end{pmatrix}$	(1,3,6,2)(4,5)
S_7	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 7 & 6 & 3 & 5 & 4 \end{pmatrix}$	(1,2)(3,7,4,6,5)
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 7 & 6 & 4 & 3 & 5 \end{pmatrix}$	(1,2)(3,7,5,4,6)
S_8	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 8 & 7 & 6 & 5 & 4 & 2 \end{pmatrix}$	(1,3,8,2)(4,7)(5,6)
S_9	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 9 & 8 & 7 & 3 & 6 & 5 & 4 \end{pmatrix}$	(1,2)(3,9,4,8,5,7,6)
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 9 & 8 & 7 & 5 & 4 & 3 & 6 \end{pmatrix}$	(1,2)(3,9,6,5,7,4,8)
S_{10}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 2 \end{pmatrix}$	(1,3,10,2)(4,9)(5,8)(6,7)
S_{11}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 1 & 11 & 10 & 9 & 8 & 3 & 7 & 6 & 5 & 4 \end{pmatrix}$	(1,2)(3,11,4,10,5,9,6,8,7)
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 1 & 11 & 10 & 9 & 8 & 6 & 5 & 4 & 3 & 7 \end{pmatrix}$	(1,2)(3,11,7,6,8,5,9,4,10)
S_{12}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 1 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 2 \end{pmatrix}$	(1,3,12,2)(4,11)(5,10)(6,9)(7,8)

The work is supported by the Grant of the President of the Russian Federation (Project MD-3952.2015.9).