On finite 5-primary groups G with disconnected Gruenberg — Kegel graph

Valeriya Kolpakova
N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. *Prime graph* (or *Gruenberg — Kegel graph*) $\Gamma(G)$ of G is defined as the graph with vertex set $\pi(G)$, in which two distinct vertices p and q are adjacent if and only if G contains an element of order pq. A group G is called *n-primary* if $|\pi(G)| = n$. We denote the number of connected components of $\Gamma(G)$ by $s(G)$, and the set of its connected components by $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$; for the group G of even order believe that $2 \in \pi_1(G)$.

Kondrat'ev determined finite almost simple 5-primary groups and their Gruenberg — Kegel graphs [1]. The author together with A. S. Kondrat'ev [2] obtained a description of chief factors of the commutator subgroups of finite non-solvable 5-primary groups G with disconnected Gruenberg-Kegel graph in the case when $G/F(G)$ is almost simple n-primary group for $n \leq 4$. Our aim is to describe 5-primary groups G with disconnected prime graph in the remaining cases. It is natural to begin the study by imposing certain restrictions on the component $\pi_1(G)$. The result of this work is describing 5-primary groups G with disconnected prime graph such that either $\pi_1(G) = \{2\}$, or $3 \notin \pi_1(G) \neq \{2\}$ and $3 \in \pi(G)$.

We prove the following two theorems. Each of the items of these theorems is realizing.

Theorem 1. *Let G be a finite 5-primary group and $\pi_1(G) = \{2\}$. Then one of the following conditions holds:*

(1) *$G \cong O(G) \rtimes S$ is Frobenius group, where $O(G)$ is 4-primary abelian group and S is cyclic 2-group or generalized quaternion group;*

(2) *G is Frobenius group with kernel $O_2(G)$ and 4-primary complement;*

(3) *$G \cong A \rtimes (B \rtimes C)$ is 2-frobenius group, where $A = O_2(G)$, B is cyclic 4-primary $2'$-group and C is cyclic 2-group;*

(4) *$G \cong L_2(r)$, $r \geq 65537$ is Mersenne or Ferma prime and $|\pi(r^2 - 1)| = 4$;*

(5) *$\overline{G} = G/O_2(G) \cong L_2(2^m)$, where either $m \in \{6, 8, 9\}$, or $m \geq 11$ is prime. If $O_2(G) \neq 1$, then $O_2(G)$ is a direct product of minimal normal subgroups of order 2^{2m} from G, each of these as $\overline{G}$-module is isomorphic to the natural $GF(2^m)SL_2(2^m)$-module;*

(6) *$\overline{G} = G/O_2(G) \cong Sz(q)$, where $q = 2^p$, $p \geq 7$ and $q - 1$ primes, $|\pi(q - \varepsilon\sqrt{2q} + 1)| = 2$ and $|\pi(q + \varepsilon\sqrt{2q} + 1)| = 1$ for $\varepsilon \in \{+, -\}$, $5 \in \pi(q - \varepsilon\sqrt{2q} + 1)$. If $O_2(G) \neq 1$, then $O_2(G)$ is a direct product of minimal normal subgroups of order q^4 from G, each of these as $\overline{G}$-module is isomorphic to the natural $GF(q)Sz(q)$-module of dimension 4.*

Theorem 2. *Let G be a finite 5-primary group with disconnected prime graph, $\overline{G} = G/F(G)$ is almost simple 5-primary group, $3 \in \pi(G)$ and $3 \notin \pi_1(G) \neq \{2\}$. Then one of the following conditions holds:*

(1) *G is isomorphic to $L_2(5^3)$ or $L_2(17^3)$;*

(2) *$G \cong L_2(p)$, where either $p \geq 65537$ is Mersenne or Ferma prime and $|\pi(p^2 - 1)| = 4$, or $p \geq 41$ is prime, $|\pi(p^2 - 1)| = 4$ and $3 \in \pi(\frac{p+1}{2})$;*

(3) *G is isomorphic to $L_2(3^r)$ or $PGL_2(3^r)$, where r is odd prime, $|\pi(3^{2r} - 1)| = 4$ and $r \notin \pi(G)$;*

(4) *$G \cong L_2(p^r)$, where $p \in \{5, 17\}$, r is odd prime, $|\pi(p^{2r} - 1)| = 4$, $3 \in \pi(\frac{p^r+1}{2})$ and $r \notin \pi(G)$.*

The work was supported by the Russian Scientific Fund (project No. 14-11-00061).

References

[1] Kondrat'ev A. S., Finite almost simple 5-primary groups и their Gruenberg-Kegel graphs. *Sib. El. Math. Rep.* **11** (2014) 634–674.

[2] Kolpakova V. A., Kondrat'ev A. S., On finite non-solvable 5-primary groups with disconnected Gruenberg — Kegel graph, such that $|\pi(G/F(G))| \leq 4$. *Fund. and app. math.*, to appear.