The chromatic number of random Cayley graphs on the symmetric group

Elena Konstantinova
Novosibirsk State University, Novosibirsk, Russia
Sobolev Institute of Mathematics, Russia
Kristina Rogalskaya
Novosibirsk State University, Novosibirsk, Russia

In 2013 Noga Alon published the first pioneer work on the chromatic number of random Cayley graphs [1]. He considered the typical behavior of the chromatic number of a random Cayley graph of a given group of order n with respect to a randomly chosen set. This behavior depends on the group. General, cyclic and abelian groups were considered by Noga Alon. As open problems, he suggested consider more accurately the case of the symmetric group Sym_n.

In this talk we investigate bichromatic Cayley graphs $\Gamma = Cay(Sym_n, S)$ on the symmetric group Sym_n with a generating set S. The necessary and sufficient conditions of a Cayley graph Γ with the chromatic number $\chi(\Gamma) = 2$ are found.

Theorem 1 *Let $\Gamma = Cay(Sym_n, S)$ is a Cayley graph on the symmetric group Sym_n. Then Γ is bichromatic if and only if the generating set S does not contain even permutations.*

The proof is based on the classical Lagrange's theorem in group theory and the Kelarev's theorem [3], which describes all finite inverse semigroups with bipartite Cayley graphs.

Theorem 2 *Let a generating set S of a random Cayley graph $\Gamma = Cay(Sym_n, S)$ consists of k randomly chosen generators of Sym_n. If $n \geqslant 2$ and $k < \frac{n!}{2}$, then $\Gamma = Cay(Sym_n, S)$ is not, asymptotically almost surely, bichromatic.*

However, these results don't give the conditions for a random Cayley graph Γ to be connected.

Open problem *What are the necessary and sufficient conditions for $\Gamma = Cay(Sym_n, S)$ to be connected, where S is a randomly chosen generating set?*

In a particular case, when the generating set S of Γ is defined by reversals, the necessary and sufficient conditions of connectedness for Γ were found by Ting Chen and Steven Skiena in [2]. Let S consists of all reversals of fixed length ℓ. Then $\Gamma = Cay(Sym_n, S)$ is connected if and only if $\ell \equiv 2 (\mathrm{mod}\ 4)$. In this case $|S| = n - \ell$ and the number of such generating sets is equal to $\lfloor \frac{n+1}{4} \rfloor$.

There are also two famous connected bichromatic Cayley graphs on the symmetric group known as the Star and the Bubble-sort graphs. These graphs are used for modelling interconnections networks [4].

The work has been supported by RFBS Grant 15-01-05867 and Grant NSh-1939.2014.1 of President of Russia for Leading Scientific Schools.

References

[1] Alon N., The chromatic number of random Cayley graphs, *European Journal of Combinatorics* **34** (2013) 1232–1234.

[2] Chen T., Skiena S. Sorting with fixed-length reversals, *Discrete applied mathematics* **71** (1996) 269–295.

[3] Kelarev A.V., On Cayley Graphs of Inverse Semigroups, *Semigroup forum* **72** (2006) 411–418.

[4] Lakshmivarahan S., Jwo J.S., Dhall S.K. Symmetry in interconnection networks based on Cayley graphs of permutation groups: a survey, *Parallel Compute.* **19** (4) (1993) 361–407.