On Cameron's question about primitive permutation groups with stabilizer of two points that is normal in the stabilizer of one of them

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Cameron formulated the following question (see [1], [3, question 9.69]). Assume that G is a primitive permutation groups on a finite set X, $x \in X$ and G_x acts regularly on the G_x-orbits $G_x(y)$ containing y (i.e. G_x induces on $G_x(y)$ a regular permutation group). Is it true that this action is faithful, i.e., that $|G_x| = |G_x(y)|$? Note that the question on the faithfulness of the action of a stabilizer G_x on a regular suborbit $G_x(y)$ was also treated earlier (see [5], [6], [7]).

It is clear that the regularity of the action of the group G_x on $G_x(y)$ is equivalent to the property $G_{x,y} \trianglelefteq G_x$, and the equality $|G_x| = |G_x(y)|$ is equivalent to the equality $G_{x,y} = 1$. Thus, Cameron's question is equivalent to the question on the fulfilment for an arbitrary primitive permutation group G on a finite set X of the following property.

(Pr) If $x \in X$ and $y \in X \setminus \{x\}$, then $G_{x,y} \trianglelefteq G_x$ implies $G_{x,y} = 1$.

Obviously, Cameron's question is also equivalent to the question on the fulfilment for an arbitrary finite group G of the following property.

(Pr*) If M_1 and M_2 are different conjugate maximal subgroups in G, then $M_1 \cap M_2 \trianglelefteq M_1$ implies $M_1 \cap M_2 \trianglelefteq G$.

In the present work (using [2]), we prove the following theorem.

Theorem. *Let G be a primitive permutation group on a finite set X and $x \in X$. Assume that the socle of G is not isomorphic to power of an exceptional group T of Lie type $E_8(q)$ with T_x of type* (d) *or* (e) *from* [4]*. Then the permutation group G satisfies property* **(Pr)**. *In particular, for such primitive groups G, the answer to Cameron's question is positive.*

References

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