On the realizability of some graphs as Gruenberg–Kegel graphs of finite groups

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We use the term "group" while meaning "finite group" and the term "graph" while meaning "undirected graph without loops and multiple edges".

Let G be a group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e., the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this graph, the vertex set is $\pi(G)$ and different vertices p and q are adjacent if and only if $pq \in \omega(G)$.

We say that a graph Γ with $|\pi(G)|$ vertices *is realizable as the Gruenberg–Kegel graph of a group G* if there exists a marking the vertices of Γ by different primes from $\pi(G)$ such that the marked graph is equal to $\Gamma(G)$. A graph Γ *is realizable as the Gruenberg–Kegel graph of a group* if Γ is realizable as the Gruenberg–Kegel graph of an appropriate group G.

The following problem arises.

Problem. *Let Γ be a graph. Is Γ realizable as the Gruenberg–Kegel graph of a group?*

Of course, in general, the problem has negative solution. For example, the graph consisting of five pairwise non-adajcent vertices (5-coclique) is not realizable as the Gruenberg–Kegel graph of a group.

In this talk, we will tell on the realizability of some graphs as Gruenberg–Kegel graphs of groups. In particular, we prove the following theorem.

Theorem. *Let Γ be a complete bipartite graph $K_{m,n}$, where $m \leq n$. Then Γ is realizable as the Gruenberg–Kegel graph of a group if and only if $m + n \leq 6$ and $(m, n) \neq (3, 3)$.*