The distribution of cycles of length $O(n)$ in the Star graph

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The Star graph $S_n = Cay(Sym_n, ST), n \geqslant 2$ is a Cayley graph on the symmetric group Sym_n with the generating set of transpositions $ST = \{t_i \in Sym_n, 2 \leqslant i \leqslant n\}$ exchanging i'th element of the permutation with the first. Graph S_n, $n \geqslant 3$, is bipartite, therefore contains only even cycles of lengths C_l, where $6 \leqslant l \leqslant n!$ [1] and has the diameter $D = \lfloor \frac{3(n-1)}{2} \rfloor$.

The current work continues the study of cyclic structure of the Star graph, started in [2], under a different approach. The distribution and the structure of vertices at each distance layer d, where $1 \leqslant d \leqslant D$, from the identity vertex is known [3]. We employ this result to study the number of cycles of lengths $2d$, $3 \leqslant d \leqslant D$, constructed from two non-intersecting shortest paths to the vertex at distance d from the identity vertex. The study of such cycles is closely related to the method proposed to solve the First Passage Percolation problem on graphs [4, 5].

Any permutation $\pi \in Sym_n$ can be represented uniquely in terms of non-intersecting cycles, i.e.

$$\pi = (1\, \pi^1_2 \dots \pi^1_{l_1})(\pi^2_1 \, \dots \, \pi^2_{l_2}) \dots (\pi^k_1 \, \dots \, \pi^k_{l_k}).$$

Denote the cycle of length l containing the element "1" as $l - CO$ and not containing it as $l - CN$, then the vertices on the distance layer d may have either

1. only a $(d + 1) - CO$;
2. an $m - CO$, $1 \leqslant m \leqslant d - 2$ and $k \geqslant 1$ items of $l_i - CN$, where $1 \leqslant i \leqslant k$, such that $d = k + (m - 1) + \sum_{i=1}^{k} l_i$.

The following theorems describe the distribution of distinct cycles in the Star graph S_n for $3 \leqslant d \leqslant D$.

Theorem 1 *The number of cycles of length $2d$ passing through the vertices with $1 - CO$ and $k \geqslant 2$ items of $l_i - CN$, over all $k + \sum_{i=1}^{k} l_i = d$, is*

$$N_{C_1} = O\left(k!(d - 3k - 2)^{4k-2} + k!(d - 3k - 2)^{3k-1}\right)(n - 1)\dots(n - d + k).$$

Theorem 2 *The number of cycles of length $2d$ passing through the vertices with $m - CO$ and $k \geqslant 2$ items of $l_i - CN$, over all $m - 1 + k + \sum_{i=1}^{k} l_i = d$, is*

$$N_{C_2} = O\left((k!)^2(d - 3k - 3)^{4k-2}\right)(n - 1)\dots(n - d + k).$$

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