Three-dimensional homogeneous spaces with invariant affine connections

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Let $(\overline{G}, M)$ be a three-dimensional homogeneous space. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. Since we are interested only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, that is two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective* if $\mathfrak{g}$ contains no non-zero ideals of $\bar{\mathfrak{g}}$, a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective.

An *isotropic* $\mathfrak{g}$-*module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that

$$x.(y+\mathfrak{g}) = [x, y]+\mathfrak{g}.$$

The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective. Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [1] with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [2]. We find all of this pairs. The curvature and torsion tensors of the invariant affine connection Λ are given by the following formulas:

$$R\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}),\ (x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto [\Lambda(x_1), \Lambda(x_2)]-\Lambda([x_1, x_2]);$$

$$T\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m},\ (x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto \Lambda(x_1)(x_2+\mathfrak{g})-\Lambda(x_2)(x_1+\mathfrak{g}) - [x_1, x_2]_{\mathfrak{m}}.$$

We restate the theorem of Wang on the holonomy algebra of an invariant connection: the Lie algebra of the holonomy group of the invariant connection defined by $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(3, \mathbb{R})$ on $(\bar{\mathfrak{g}}, \mathfrak{g})$ is given by

$$V + [\Lambda(\bar{\mathfrak{g}}), V] + [\Lambda(\bar{\mathfrak{g}}), [\Lambda(\bar{\mathfrak{g}}), V]] + \dots,$$

where V is the subspace spanned by $\{[\Lambda(x), \Lambda(y)] - \Lambda([x, y])|x, y \in \bar{\mathfrak{g}}\}$.

We describe all local three-dimensional homogeneous spaces, allowing affine connections, it is equivalent to the description of effective pairs of Lie algebras, and all invariant affine connections on the spaces together with their curvature, torsion tensors and holonomy algebras. We use the algebraic approach for description of connections, methods of the theory of Lie groups, Lie algebras and homogeneous spaces.

The results of work can be used in research work on the differential geometry, differential equations, topology, in the theory of representations, in the theoretical physics. In particular, the results can find practical application in general theory of relativity, which, with mathematical point of view, is based on the geometry of the curved spaces, in the nuclear physics and physics of elementary particles that are associated with geometric interpretation of equations. Methods stated in the work, can be applied for the analysis of physical models, and algorithms classification of homogeneous spaces, affine connections on these spaces, curvature and torsion tensors, holonomy algebras can be computerized and used for the decision of similar problems in large dimensions.

References

[1] K. Nomizu, Invariant affine connections on homogeneous spaces. *Amer. J. Math.* **76** (1954) 33-65.

[2] S. Kobayashi, K. Nomizu, Foundations of differential geometry. *John Wiley and Sons, New York.* **1** (1963); **2** (1969).