Formations of finite groups and Hawkes graph

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All groups considered are finite. There have been a lot of papers recently in which with every finite group associates certain graph. The considered problem was to analyze the relations between the structure of a group and the properties of its graph. This trend goes back to 1878 when A. Cayley [1] introduced his graph.

Let $\pi(G)$ be the set of prime divisors of $|G|$. Recall [2] that the Gruenberg-Kegel or the prime graph Γ_p of a group G is the graph with the vertex set $\pi(G)$ and (p,q) is an edge if and only if G contains element of order pq. This graph is connected to the problem of recognition of groups by their graph. Recall that a group G is called recognizable by the prime graph if $\Gamma_p(G) = \Gamma_p(H)$ implies $H \simeq G$ for any group H. There are many non-isomorphic groups with nontrivial solvable radical and the same prime graph. That is why of prime interest (for example see [3]) is this problem only for simple and almost simple groups. In this paper we will consider the recognition problem up to a class of groups.

Definition 1. A function $\Gamma : \{\text{groups}\} \to \{\text{graphs}\}$ is called graph function.

Definition 2. Let Γ be a graph function and $\mathfrak{X}$ be a class of groups. We shall say that $\mathfrak{X}$ is recognized by Γ if from $G_1 \in \mathfrak{X}$ and $\Gamma(G_1) = \Gamma(G_2)$ it follows that $G_2 \in \mathfrak{X}$.

Problem 1. (a) Let Γ be a graph function. Describe all group classes (formations, Fitting classes, Schunk classes) that are recognizable by Γ.
(b) Let $\mathfrak{X}$ be a class of groups (formation, Fitting class, Schunk class). Find graph functions Γ that recognize $\mathfrak{X}$.

T. Hawkes [4] in 1968 considered a directed graph of a group G whose set of vertices is $\pi(G)$ and (p,q) is an edge if and only if $q \in \pi(G/O_{p',p}(G))$. In particular he showed that a group G has a Sylow tower for some linear order ϕ if and only if its graph has not got circuits. We shall call this graph Hawkes graph and will denote it $\Gamma_H(G)$.

Theorem 1. *Let $\mathfrak{F}$ be a formation of groups. Then $\mathfrak{F}$ is recognized by Γ_H if and only if $\mathfrak{F} = LF(f)$ is a local formation where f is formation function defined as follows: $f(p) = \mathfrak{G}_{f(p)}$ if $p \in \pi(\mathfrak{F})$ and $f(p) = \emptyset$ otherwise.*

References

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[3] A. V. Vasil'ev, I. B. Gorshkov. On recognition of finite simple groups with connected prime graph. *Sib. Mat. J.* **50**(2) (2009), 233–238.

[4] T. Hawkes, On the class of the Sylow tower groups. *Math. Z.* **105** (1968), 393–398.