On the pronormality and strong pronormality of Hall subgroups

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Throughout a set of primes is denoted by π. A subgroup H of G is called a *π-Hall subgroup*, if H is a π-group (i.e. all its prime divisors are in π), while the index of H is not divisible by primes from π. A subgroup is said to be a *Hall subgroup* if it is a π-Hall subgroup for some set of primes π. A subgroup H of G is called *pronormal*, if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

In Kourovka Notebook recorded the next problem [1, 18.32]: is every Hall subgroup of a finite group pronormal in its normal closure? The negative solution gives by the following

Theorem. *Let a set of primes π be such that*

(1) *there exists a simple group X which contains more than one class of conjugated π-Hall subgroups;*

(2) *there exists a simple group Y such that it contains a π-Hall subgroup which is not equal to self normalizer in Y.*

Thus in the regular wreath product $G = X \wr Y$ exists a not pronormal π-Hall subgroup, normal closure of which is equal to G.

For example, set $\{2, 3\}$ satisfies theorem conditions: group $X = L_3(2)$ contains two classes of conjugated $\{2, 3\}$-Hall subgroups and group $Y = L_2(16)$ contains $\{2, 3\}$-Hall subgroup which is not equal to self normalizer in Y.

A subgroup H of G is called *strongly pronormal*, if, for each $K \leq H$ and every $g \in G$, the subgroup K^g is conjugate with a subgroup of H (but not necessary with K) by an element from $\langle H, K^g \rangle$.

Also a negative solution of the problem [1, 17.45(б)] issue was obtained: in a finite simple group, are Hall subgroups always strongly pronormal?

More specifically, it was shown that $S_{10}(7)$ contains a $\{2, 3\}$-Hall subgroup, which is not strongly pronormal. Note that there are not known examples of pronormal Hall subgroups which are not strongly pronormal before.

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References

[1] Kourovka notebook; Edition 18, Novosibirsk 2014.