Perfect k-colorings of infinite circulant graphs with a continuous set of distances

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Consider an infinite graph $Ci_\infty(d_1, d_2, d_3, ..., d_n)$, whose set of vertices is the set of integers, and two vertices are adjacent if they are on the distance $d \in \{d_1, d_2, d_3, ..., d_n\}$. Let us call it an *infinite circulant graph*. Also we consider a finite graph $Ci_t(d_1, d_2, d_3, ..., d_n)$ with the set of vertices coinciding with the set Z_t and for each vertex v the multiset of incident edges is $\{(v, v + d_i \text{ mod t})|i = 1, 2, ..., n\}$. There is a natural homomorphism from the set of vertices of the graph $Ci_\infty(d_1, d_2, d_3, ..., d_n)$ on the set of vertices of the graph $Ci_t(d_1, d_2, d_3, ..., d_n)$ corresponding with the homomorphism from Z to Z_t.

Let k be a positive integer. A *k-coloring* of vertices of a graph $G = (V, E)$ is a map $\varphi : V \to \{1, 2, ..., k\}$. If $\varphi(v) = s$ for some vertex v, then s is the *color* of v.

A k-coloring of vertices is called *perfect*, if for each $i, j = 1, 2, ..., k$ are not necessarily different there is an uniquely defined non-negative integer α_{ij} which is equal to the number of vertices of the color j in the neighborhood of each vertex of the color i. The *period* T of a coloring is a sequence $\gamma_1\gamma_2...\gamma_r$, where $\gamma_i = \varphi(v_{m+i})$ for some number m, and $\varphi(v_l) = \varphi(v_{l+jr})$ for every l and j. The number r is the *length* of the period T. It is clear that the coloring of a regular graph is uniquely defined by its period.

Perfect 2-colorings of circulant graphs are considered in [1, 2]. We are interested in so-called circulant graphs with a continuous set of distances, i.e. in ones with the property $d_i = i, i = 1, 2, 3, ..., n$. The fool description of 2-colorings of graphs $Ci_\infty(n) = Ci_\infty(1, 2, ..., n)$ for an arbitrary positive integer n is given in [2]. A description of colorings with k colors for $k \geq 3$ presents severe difficulties, in particular, the natural homomorphism from n-dimensional grid Z^n on $Ci_\infty(n)$ shows that the problem is rather complicate.

Here we present the main result:

Theorem Let k, n be positive integers. The set of perfect colorings of a graph $Ci_\infty(n)$ contains all perfect colorings of graphs $Ci_t(n)$ for $t = 2n, 2n + 1, 2n + 2$ and the following ones:
1. $123...k$;
2. $123...(k-1)k(k-1)...32$;
3. $123...(k-1)kk(k-1)...32$;
4. $123...(k-1)kk(k-1)...321$.

It should be noted that last four colorings in the theorem are perfect for every n.

We conjecture that there are no other perfect colorings of the $Ci_\infty(n)$.

References

[1] D. B. Khoroshilova, On circular perfect two-color colorings. (Russian). *Diskretn. Anal. Issled. Oper.* **16**(1) (2009) 80-92.

[2] O. G. Parshina, Perfect 2-colorings of infinite circulant graphs with continuous set of distances , *Journal of Applied and Industrial Mathematics.* **8**(3) (2014) 357-361.