On some subgroups of finite products of generalized nilpotent groups

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All groups considered are finite. Let a group $G = AB$ be a product of two its subgroups A and B. A subgroup H of $G = AB$ is called prefactorized if $H = (A \cap H)(B \cap H)$, it is called factorized [1] if, in addition, H contains the intersection $A \cap B$. For a saturated formation Heineken [2], for a Schunck class $\mathfrak{X}$ Amberg and Höfling [3] investigated prefactorized and factorized $\mathfrak{X}$-maximal subgroups (in particular $\mathfrak{X}$-projectors) of the group $G = AB$ with nilpotent subgroups A and B (see [4, 3.2.20, 3.2.22]).

We use notations and definitions from [5], [6]. Let π be a set of primes and π' the complement to π in the set of all primes. A group G is called π-decomposable if $G = G_\pi \times G_{\pi'}$ and a Hall π-subgroup G_π is nilpotent. The set of distinct primes dividing $|G|$ is denoted by $\pi(G)$. A non-empty homomorph $\mathfrak{X}$ is a Schunck class if any group G, all of whose primitive factor groups are in $\mathfrak{X}$, is itself in $\mathfrak{X}$. If $\mathfrak{H}$ and $\mathfrak{X}$ are classes of groups then $\mathfrak{H}\mathfrak{X} = (G|G$ has a normal subgroup $N \in \mathfrak{H}$ with $G/N \in \mathfrak{X})$. $\mathfrak{G}_{\pi'}$ denotes the class of all π'-groups.

Theorem. *Let $\mathfrak{X}$ be a class of groups and $\mathfrak{X} = \mathfrak{G}_{\pi'}\mathfrak{X}$. Let G be a π-soluble group and $G = AB$ be a product of two π-decomposable subgroups A and B.*

1) *If $\mathfrak{X}$ is a Schunck class such that $\pi(A) \cap \pi(B) \subseteq Char(\mathfrak{X})$, then every $\mathfrak{X}$-maximal subgroup of G has a factorized conjugate.*

2) *If $\mathfrak{X}$ is a saturated formation, then every $\mathfrak{X}$-maximal subgroup of G has a prefactorized conjugate.*

Recall that a subgroup H of a group G is an $\mathfrak{X}$-projector if HN/N is $\mathfrak{X}$-maximal in G/N for every normal subgroup N of G. If $\mathfrak{X}$ is a Schunck class and $\mathfrak{X} = \mathfrak{G}_{\pi'}\mathfrak{X}$ then every π-soluble group G has an $\mathfrak{X}$-projector and any two $\mathfrak{X}$-projectors of G are conjugate [7].

Corollary. *Let $\mathfrak{X}$ be a class of groups and $\mathfrak{X} = \mathfrak{G}_{\pi'}\mathfrak{X}$. Let G be a π-soluble group and $G = AB$ be a product of two π-decomposable subgroups A and B.*

1) *If $\mathfrak{X}$ is a Schunck class such that $\pi(A) \cap \pi(B) \subseteq Char(\mathfrak{X})$, then G has a unique factorized $\mathfrak{X}$-projector.*

2) *If $\mathfrak{X}$ is a saturated formation, then G has a unique prefactorized $\mathfrak{X}$-projector.*

The example 1 [3] shows that the condition $\pi(A) \cap \pi(B) \subseteq Char(\mathfrak{X})$ of theorem can not be discarded.

References

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