On arc-transitive distance-regular covers of complete graphs related to $SU_3(q)$

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In 1991, P.J. Cameron has discovered a family of arc-transitive distance-regular covers of complete graphs, which are obtained by the following construction proposed in [3, p.90]. Let E be the quadratic extension of the finite field F of q elements. Denote by V the 3-dimensional vector space over E equipped with a non-degenerate Hermitian form B. Let U be a subgroup of E^* of index r. Let Ψ_r be the graph on the set of U-orbits on the isotropic vectors of V with two vertices vU and wU being adjacent if and only if $B(v,w) = 1$. By [3, Proposition 5.1 (iv)] Ψ_r is distance-regular (with intersection array $\{q^3,(r-1)(q^2-1)(q+1)/r,1;1,(q^2-1)(q+1)/r,q^3\}$) if and only if either q is even and r divides $q+1$, or q is odd and r divides $(q+1)/2$. The question naturally arises whether this family comprises (up to isomorphism) all distance-regular covers of complete graphs with the antipodality index dividing $q+1$, which possess an arc-transitive automorphism group, isomorphic to $SU_3(q)$. As we will show below, it turns out, that the answer is negative.

Let $G = SU_3(q)$ denote the special unitary group on V and put $K = G_{\langle e_1\rangle,\langle e_2\rangle}$, where e_1 and e_2 are two non-collinear isotropic vectors of V. Take P to be the subgroup of K of order $q-1$, and let S be the subgroup of $G_{\langle e_1\rangle}$ of order q^3. Put $H = SP$. Assume that g is a 2-element of G interchanging $\langle e_1\rangle$ with $\langle e_2\rangle$ such that $g^2 \in H$. Let $\Gamma(G,H,HgH)$ denote the graph with vertex set $\{Hx \mid x \in G\}$ whose edges are the pairs $\{Hx,Hy\}$ such that $xy^{-1} \in HgH$.

Theorem. *If q is odd, then $\Gamma(G,H,HgH)$ is distance-regular if and only if g is an element of order 4, while if q is even, then g is an involution and $\Gamma(G,H,HgH)$ is a distance-regular graph isomorphic to Ψ_{q+1}. In both cases, the resulting distance-regular graph has intersection array $\{q^3,q(q^2-1),1;1,q^2-1,q^3\}$, does not depend on the choice of the element g (of the given order) and admits distance-regular quotients with intersection array $\{q^3,(i-1)(q^2-1)(q+1)/i,1;1,(q^2-1)(q+1)/i,q^3\}$ for each proper divisor i of $q+1$.*

Remark. Assume that q is odd and let g be of order 4. Distance-regularity of $\Gamma(G,H,HgH)$ appear to be first shown in the course of this work. Note that if γ is an element of E^* such that $\gamma^q = -\gamma$ and $U = F^*$, then $\Gamma(G,H,HgH)$ is isomorphic to the graph Φ on the set of U-orbits on the isotropic vectors of V with two vertices vU and wU being adjacent if and only if $B(v,w) \in U\gamma$. The construction of the graph Φ fits in the construction described in [2, Proposition 12.5.4], which generalizes the Cameron construction. However, the case $r = q+1$ for an odd q has not been completely considered in [2]. Note also, that if in definition of Φ we assume $\gamma \in U$ instead of the condition $\gamma^q = -\gamma$, then we get $\Phi \simeq \Psi_{q+1} \simeq \Gamma(G,H,HgH)$ for an involution g.

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