On finite groups with submodular Sylow subgroups

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Throughout these abstracts, all groups are finite. Recall that a subgroup M of a group G is called modular in G, if the following hold:
1) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \le G, Z \le G$ such that $X \le Z$, and
2) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \le G, Z \le G$ such that $M \le Z$.
Note that a modular subgroup is a modular element (in the sense of Kurosh [1, Chapter 2, p. 43]) of a lattice of all subgroups of a group. Properties of modular subgroups were studied in the book [1]. Groups with all subgroups are modular were studied by R. Schmidt [1], [2] and I. Zimmermann [3]. By parity of reasoning with subnormal subgroup, in [3] the notion of a submodular subgroup was introduced.

Definition [3]. *A subgroup H of a group G is called submodular in G, if there exists a chain of subgroups $H = H_0 \le H_1 \le \ldots \le H_{s-1} \le H_s = G$ such that H_{i-1} is a modular subgroup in H_i for $i = 1, \ldots, s$.*

It's well known that in a nilpotent group every Sylow subgroup is normal (subnormal). In the paper [3] groups with submodular subgroups were studied. In particular, it was proved that in a supersoluble group G every Sylow subgroup is submodular if and only if $G/F(G)$ is abelian of squarefree exponent. A criterion of the submodularity of Sylow subgroups in an arbitrary group was found.

We continue study of groups with submodular Sylow subgroups. A group we call strongly supersoluble and denote $s\mathfrak{U}$, if it is supersoluble and every Sylow subgroup is submodular in it. Denote $\mathfrak{V}$ a class of all abelian groups of exponent free from squares of primes; $sm\mathfrak{U} = (\ G \mid$ every Sylow subgroup of the group G is submodular in $G\ )$.

We obtained the following results:

Theorem 1. *Let G be a group. Then the following hold:*
1) *if $G \in sm\mathfrak{U}$ and $H \le G$, then $H \in sm\mathfrak{U}$;*
2) *if $G \in sm\mathfrak{U}$ and $N \trianglelefteq G$, then $G/N \in sm\mathfrak{U}$;*
3) *if $N_i \trianglelefteq G$ and $G/N_i \in sm\mathfrak{U}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in sm\mathfrak{U}$;*
4) *if $H_i \in sm\mathfrak{U}$, $H_i \trianglelefteq G$, $i = 1, 2$ and $H_1 \cap H_2 = 1$, then $H_1 \times H_2 \in sm\mathfrak{U}$;*
5) *if $G/\Phi(G) \in sm\mathfrak{U}$, then $G \in sm\mathfrak{U}$;*
6) *the class of groups $sm\mathfrak{U}$ is a hereditary saturated formation.*

Theorem 2. *The class of all groups with submodular Sylow subgroups is a local formation and has a local screen f such that $f(p) = (G \in \mathfrak{S} \mid \mathrm{Syl}(G) \subseteq \mathfrak{A}(p-1) \cap \mathfrak{V})$ for every prime p.*

Theorem 3. *Let G be a group. Then the following statements are equivalent:*
1) *every Sylow subgroup is submodular in G;*
2) *G is Ore dispersive and every its biprimary subgroup is strongly supersoluble;*
3) *every metanilpotent subgroup of G is strongly supersoluble.*

References

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[3] Zimmermann, I. Submodular Subgroups in Finite Groups. *Math. Z.* **202** (1989) 545–557.