Symmetry via graphs and their groups of automorphisms
Lecture 3: On the full automorphism group in VT graphs – from even to odd automorphisms

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VERTEX-TRANSITIVE GRAPHS OF ORDER $2p$

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Introduction and Brief History

In the past ten years there has been a considerable amount of activity in the area of *circulant graphs* and *digraphs*. Most of this has consisted of investigation of basic properties of circulants along with some applications. We shall now summarize some of this activity.

Let $S \subseteq \{1, 2, \ldots, n-1\}$ with the property that $i \in S$ implies $n - i \in S$. The *circulant graph* $G(n, S)$ is the graph with vertex set $v_0, v_1, \ldots, v_{n-1}$, and an edge joining v_i and v_j if and only if $j - i \in S$, where we take $j - i$ modulo n; S is called the *symbol* of $G(n, S)$. We may remove the restriction, $i \in S$ implies $n - i \in S$, and define the *circulant digraph* $G(n, S)$ as the digraph with vertex set $v_0, v_1, \ldots, v_{n-1}$ and an arc from v_i to v_j if an only if $j - i \in S$ modulo n. (All subscript arithmetic throughout this paper shall be modulo n, henceforth we refrain from mentioning it.)

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Description of 2-Circulants

Yap [12] gives a method of constructing vertex-transitive graphs of order $2p$, but this method does not produce all such graphs. The following construction also produces vertex-transitive graphs of order $2p$. We know of no vertex-transitive graph with $2p$ vertices that cannot be so constructed.

Description of VT graphs of order $2p$ with p a prime

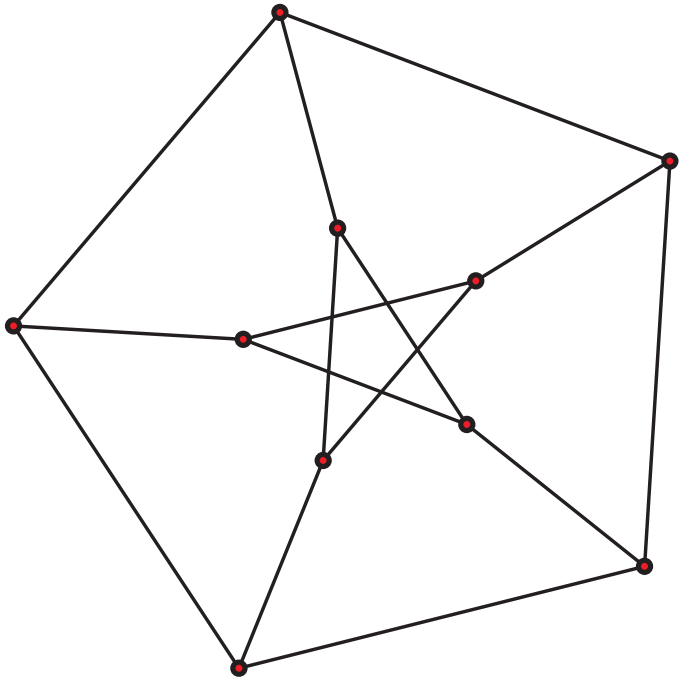
- Then the graph is a p-bicirculant, that is, a graph with a $(2, p)$-semiregular automorphism.
  An element of a permutation group is (m, n)-semiregular if it has m orbits of size n and no other orbit.
- A graph admitting an (m, n)-semiregular automorphism is called an (m, n)-multicirculant (a.k.a. (m, n)-galactic graph).

Description of VT graphs of order $2p$ with p a prime

Let ρ be a $(2, n)$-semiregular automorphism of an n-bicirculant X, let U and W be the two orbits of ρ, and let $u \in U$ and $w \in W$.

Let $R = \{r \in Z_n \setminus \{0\} \mid u \sim \rho^r(u)\}$ be the symbol of the n-circulant induced on U and, let S be the symbol of the n-circulant induced on W (relative to ρ). Moreover, let $T = \{t \in Z_n \mid u \sim \rho^t(w)\}$. The ordered triple $[R, S, T]$ is the symbol of X relative to (ρ, u, w). Note that $R = -R$ and $S = -S$ are symmetric, that is, inverse-closed subsets of $Z_n \setminus \{0\}$ and are independent of the choice of vertices u and w.

Description of VT graphs of order $2p$ with p a prime

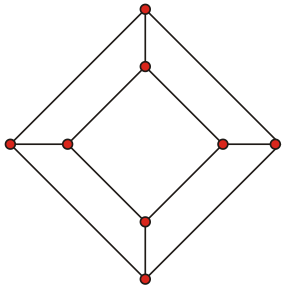


The Petersen graph
$R = \{\pm 1\}$, $S = \{\pm 2\}$, $T = \{0\}$.

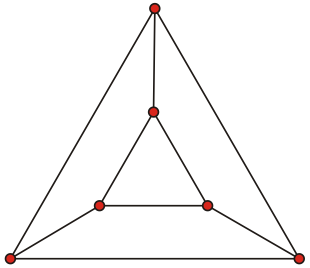
Description of VT graphs of order $2p$ with p a prime

In VT n-bicirculant either

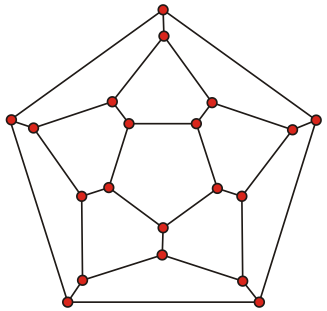
- $\exists$ a swap, an automorphism interchanging the two orbits;
- $\exists$ a mixer, an automorphism mixing the two orbits;
- $\exists$ a swap and a mixer.



both



swap



mixer

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X VTG of order $2p$, $G \leq \mathrm{Aut}X$ transitive

- G is imprimitive, blocks of size p;
- G is imprimitive, blocks of size 2,
- G is primitive.

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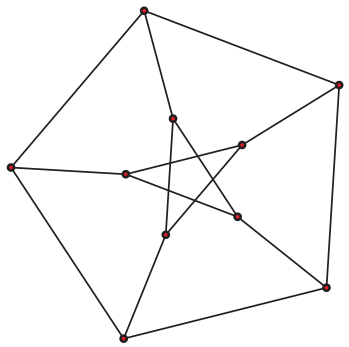
- G is imprimitive, blocks of size 2 – a swap exists as a consequence of existence of a smaller $H \leq G$ with blocks of size p (DM'81),
- G is primitive – a swap exists as a consequence of the classification of finite simple groups (CFSG).

G is primitive

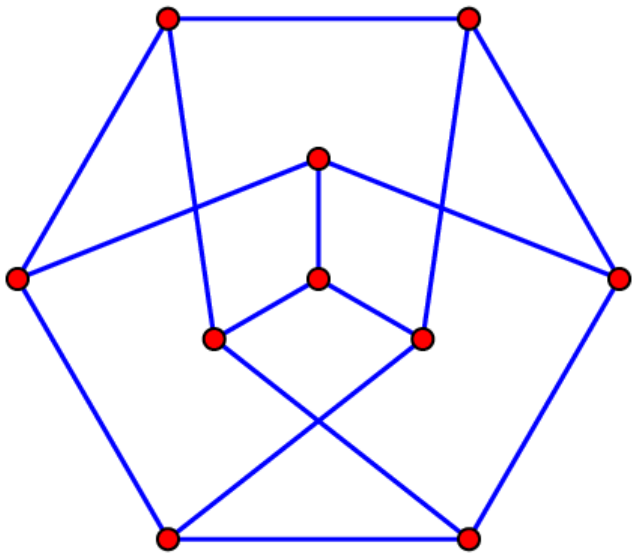
Example:
$p = 5$, A_5, S_5 acting on pairs from $\{1, 2, 3, 4, 5\}$.
Associated graphs: the Petersen graph and its complement.

By CFSG any primitive group of degree $2p$, $p > 5$, is 2-transitive.

No CFSG-free proof of this fact exists.
No CFSG-free answer to an even simpler question regarding (existence of odd automorphisms in) the full automorphism group.



STOA – Symmetry Through Odd Automorphisms



The full automorphism group of the Petersen graph contains involutions with three orbits of size 2 and four fixed vertices, and hence odd (as permutations) automorphisms.

STOA – Symmetry Through Odd Automorphisms

A crucial question in algebraic graph theory and beyond:

Given a graph, are there any symmetries beyond the obvious ones, and, if yes, how can one determine the full set?

We approach this question by building on the duality of even/odd permutations associated with graphs.

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- When H consists of even permutations only, is there a group G containing odd automorphisms of X and H as a subgroup? *Clearly, the examples of the Petersen graph and the 3-cube tell us that the answer is not always yes.*
- Further, what is the minimum index $[G : H]$ among all such groups G? *In the Petersen graph $H = A_5$ is the only transitive group of even permutations and its only "extension" is $G = S_5$, and so its minimal index is 2 as one would hope for. However, an odd "extension" G of $H = \langle(1\,2\,3\,4\,5)\rangle$ acting on the complete graph K_5 is either of order 20 or S_5, and the corresponding indices are 4 or 24.*

Cubic arc-transitive graphs:

s	Type	Bipartite?	s	Type	Bipartite?	s	Type	Bipartite?
1	$\{1\}$	Sometimes	3	$\{2^1, 3\}$	Never	5	$\{1, 4^1, 4^2, 5\}$	Always
2	$\{1, 2^1\}$	Sometimes	3	$\{2^2, 3\}$	Never	5	$\{4^1, 4^2, 5\}$	Always
2	$\{2^1\}$	Sometimes	3	$\{3\}$	Sometimes	5	$\{4^1, 5\}$	Never
2	$\{2^2\}$	Sometimes	4	$\{1, 4^1\}$	Always	5	$\{4^2, 5\}$	Never
3	$\{1, 2^1, 2^2, 3\}$	Always	4	$\{4^1\}$	Sometimes	5	$\{5\}$	Sometimes
3	$\{2^1, 2^2, 3\}$	Always	4	$\{4^2\}$	Sometimes			

$K_4 = \{1, 2^1\}$, $K_{3,3} = \{1, 2^1, 2^2, 3\}$, $Q_3 = \{1, 2^1\}$, $F010A = \{2^1, 3\}$,
$F014A = \{1, 4^1\}$, $F016A = \{1, 2^1\}$, $F018A = \{1, 2^1, 2^2, 3\}$,
$F020A = \{1, 2^1\}$, $F020B = \{2^1, 2^2, 3\}$

STOA – Symmetry Through Odd Automorphisms

Theorem: *Let X be a cubic arc-transitive graph of order $2n$. Then Table below gives a full information on existence of odd automorphisms in X.*

Type	Odd automorphisms exist if and only if
$\{1\}$	n odd
$\{1, 2^1\}$	n odd, or $n = 2^{k-1}(2t+1)$ and X is a $(2t+1)$-Cayley graph on a cyclic group of order 2^k, where $k \geq 2$
$\{2^1\}$	n odd and X bipartite
$\{2^2\}$	never
$\{1, 2^1, 2^2, 3\}$	n odd
$\{2^1, 2^2, 3\}$	n odd
$\{2^1, 3\}$	n odd
$\{2^2, 3\}$	n odd
$\{3\}$	n odd and X bipartite
$\{1, 4^1\}$	n odd
$\{4^1\}$	n odd and X bipartite
$\{4^2\}$	n odd
$\{1, 4^1, 4^2, 5\}$	n odd
$\{4^1, 4^2, 5\}$	n odd
$\{4^1, 5\}$	never
$\{4^2, 5\}$	never
$\{5\}$	n odd and X bipartite

Thank you!