MONSTROUS MOONSHINE

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(Very brief) History

✓ **1979** [L.H.Conway & S.P.Norton. Monstrous Moonshine. Bull. London Math. Soc. 11: 308 – 339] Unexpected relationships between finite simple groups and modular functions, Monster group and modular forms, were pointed out. It was conjectured a relation of conjugacy classes of the Monster and the action of the certain subgroups of $SL_2(\mathbb{R}) \curvearrowright H = \{z \in \mathbb{C} | Im\, z > 0\}$. This implies that the action $SL_2(\mathbb{R}) \curvearrowright H$ contains information about representations of the Monster.

Monstrous Moonshine is a collection of questions and (less) answers inspired by these observations.

✓**1988** [I.Frenkel, J.Lepowsky, & A.Meurman. Vertex Operator Algebras and the Monster. Acad. Press, NY] Explanation how the Monster is related to modular forms: it acts on appropriate vertex algebra $V^\natural$ by its automorphisms. The tool to handle the problem posed by Conway and Norton.

✓ **1992** [R.E.Borcherds. Monstrous moonshine and monstrous Lie superalgebras. Invent. Math. 109: 405 – 444] Proof of the main conjecture of Conway and Norton based on a special class of infinite dimensional Lie algebras – so called generalized Kac – Moody algebras – and the No-Ghost theorem from the string theory. Also Borcherds pointed out other remarkable connections among sporadic simple groups and modular forms. In 1998 he was awarded a Fields medal for his contributions.

Principally, Moonshine discovers two interplays:

$$\begin{array}{rcl} \text{Algebra} & \Leftrightarrow & \text{Modular stuff} \\ \text{Mathematics} & \Leftrightarrow & \text{Physics} \end{array}$$

Groups and actions

Denote
$GL_2(\mathbb{R})^+ = \{\alpha \in GL_2(\mathbb{R}) | \det \alpha > 0\}$, $SL_2(\mathbb{R}) = \{\alpha \in GL_2(\mathbb{R}) | \det \alpha = 1\}$.

$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a Riemann sphere

Action $GL_2(\mathbb{R}) \curvearrowright \overline{\mathbb{C}}$ by linear fractional transformations is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

Upper half-plane $H = \{z \in \mathbb{C} | Im\, z > 0\}$ is $GL_2(\mathbb{R})$-invariant.

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = z$ then $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\} \curvearrowright H$.

$\Gamma := PSL_2(\mathbb{Z})$ **modular group**.

Generators and fundamental domain for Γ

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $Tz = z + 1 \;\; \Rightarrow$ each $SL_2(\mathbb{Z})$-orbit intersects $\{z \in H | -1/2 \leq Re\, z \leq 1/2\}$.

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $Sz = -1/z \;\; \Rightarrow z$ and $-1/z$ are in the same orbit.

Each orbit intersects $\{z \in H | |z| \geq 1\}$. If $|z| = 1$ then $Sz = -1/z = -\overline{z}$.

$D = \{z \in H | -1/2 \leq Re\, z \leq 1/2, |z| \geq 1\}$ a **fundamental domain** for $\Gamma = PSL_2(\mathbb{Z})$. It intersects each orbit at just one point.

Formally,
a connected subset $D \subset H$ is a **fundamental domain** of a discrete subgroup G of $SL_2(\mathbb{R})$ if
(1) $H = \bigcup_{\gamma \in \Gamma} \gamma D$; (2) if $U = int\, D$ then $D = clos\, U$; (3) $\forall \gamma \in G \;\gamma U \cap U = \emptyset$.

It can be proven that $\Gamma =< T, S >$.

Quotient

The map $D \to H/SL_2(\mathbb{Z})$ is surjective and its restriction on $int\, D$ is injective.
$H/SL_2(\mathbb{Z})$ is a Riemann surface of genus 0 with one point removed. There is an isomorphism $H/SL_2(\mathbb{Z}) \cong \mathbb{C}$. It extends to

$$H/SL_2(\mathbb{Z}) \cup \{i\infty\} \cong \mathbb{C} \cup \infty = \mathbb{CP}^1 : i\infty \mapsto \infty. \tag{1}$$

Such an isomorphism is not unique. If j is one of isomorphisms (1) then another is $a(j+b)$ for $a \neq 0, b \in \mathbb{C}$.
Since for any $z \in H$ points z and $z+1$ are in the same orbit than
$j(z) = j(z+1) \;\Rightarrow j$ has a Fourier expansion $j(z) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n z}$, or $j(z) = \sum_{n\in\mathbb{Z}} c_n q^n$ for $q = e^{2\pi i z}$.

Modular Functions, Modular Forms

Meromorphic function $f : H \to \mathbb{C}$ is **modular of weight** $2k$, $k \in \mathbb{Z}$, if for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z).$$

If f is also holomorphic everywhere (and at ∞) then f is **modular form**.
Example 1. The **Eisenstein series**

$$E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + \ldots,$$

where $\sigma_3(n) = \sum_{d|n} d^3$, is a modular form of weight 4.
Example 2. The **Dedekind function**

$$\eta(z) = q \prod_{n\geq 1} (1-q^n)^{24} = q - 24q^2 + 252q^3 - \ldots$$

is modular form of weight 12.
Example 3. The **main modular function**, or j**-function**, or **Hauptmodul**

$$j(z) = \frac{E_4(z)^3}{\eta(z)} = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \ldots$$

is a modular function of weight 0.
It is constant on orbits of $SL_2 \curvearrowright H$, holomorphic on H and has simple pole at ∞ $(q=0) \Rightarrow$ it gives holomorphic isomorphism $\overline{H}/SL_2(\mathbb{Z}) \xrightarrow{\sim} \overline{\mathbb{C}}$, where $\overline{H} = H \cup \mathbb{Q} \cup \{i\infty\}$.

The $SL_2(\mathbb{Z})$-orbits of points sitting in $\mathbb{Q} \cup \{i\infty\}$ are called **cusps** and their role is to compactify the punctured Riemann sphere $H/SL_2(\mathbb{Z})$. There are much fewer meromorphic functions living on compact manifolds rather then of functions living on their non-compact subsets.

Since any other isomorphism has a form $a(j(z)+b)$ with constants $a \neq 0, b$, then define **normalized Hauptmodul,** or **canonical isomorphism** as $J(z) = j(z) - 744 = q^{-1} + 196\,884q + 21\,493\,760q^2 + \ldots$.

Kleinian groups

Set $\Gamma := SL_2(\mathbb{Z})$, define **principal congruence subgroups**

$$\Gamma(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$$

and the class

$$\Gamma_0(n) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 mod\, n\}$$

Fricke [Fricke R. Die Elliptische Functionen und Ihrer Anwendungen. Teubner, Leipzig, 1922] investigated surfaces associated with $\Gamma_0(N)$. For N prime, $\Gamma_0(N)$ lead to genus 0 surfaces iff $p-1|24$.

The **Fricke involution** $z \mapsto -1/Nz$ leads to the group

$$\Gamma_0(N)^+ = \left\langle \Gamma_0(N), \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \right\rangle.$$

The normalizer $N(\Gamma_0(N))$ of $\Gamma_0(N)$ in $PSL_2(\mathbb{R})$ was described by Atkin and Lehner [Atkin A.L. & Lehner J. Hecke operators on $\Gamma_0(N)$. Math. Ann, 185: 134 – 160, 1970]. When $N = p$ prime, $N(\Gamma_0(N)) = \Gamma_0(n)^+$.
For a prime p, $\Gamma_0(N)^+$ has a genus 0 property iff

$$p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71.$$

(The proof of Fricke was completed by Ogg [Ogg A.P. Automorphismes des courbes modulaires. Sem. Delange – Pisot – Poitou, 16e année, 7,1975])
This set of primes is exactly the set of prime divisors of the order of Monster Group. Its existence was only conjectured (by Fischer and Griess) but still not proven at that time.

Finite simple groups
A group G is **simple** if it has no nontrivial proper normal subgroups.
Every finite simple group is isomorphic to one of the following:
- a cyclic group $\mathbb{Z}_p$ of prime order p;
- an alternating group A_n, $n \geq 5$;
- a simple group of Lie type over a finite field, i.e. $PSL_n(\mathbb{F}_{p^m})$;
- some one of the 26 sporadic simple groups.

The largest of sporadic simple groups is the **Monster group** $\mathbb{M}$. It contains among its subquotients twenty of the sporadic simple groups (so called **Happy Family**) except for **pariah** $J_3, Ru, O'N, Ly, J_4$. The existence and basic properties of the Monster group as the largest of sporadic groups, were predicted independently by Fischer and Griess in 1973.

Its order equals

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

Monster has 194 conjugacy classes and irreducible characters.
The degrees of smallest irreducible characters of $\mathbb{M}$ are

$$d_0 = 1, \quad d_1 = 196\,883, \quad d_2 = 21\,296\,876, \quad d_3 = 842\,609\,326, \ldots.$$

The character table of $\mathbb{M}$ was determined by Fischer, Livingstone and Thorne in 1978.

John McKay observation:

$$\begin{array}{cccccccccc} J(q) = & 1 & \cdot q^{-1} & + & 196\,884 & \cdot q & + & 21\,493\,760 & \cdot q^2 & + \ldots \\ & \| & & & \| & & & \| & & \\ & d_0 & & & d_1 + d_0 & & & d_2 + d_1 + d_0 & & \end{array}$$

Also the number 196 884 appears as the dimension of the Griess algebra (to be explained later) for $\mathbb{M}$. In [Thompson J. Some numerology between the Fischer – Griess monster and modular functions. Bull. London Math. Soc. 11: 352 – 353, 1979] there are some other relations between the coefficients c_i of the Fourier expansion for normalized Hauptmodul

$$\begin{aligned} &J(z) = q^{-1} + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 \\ &+20\,245\,856\,256q^4 + 333\,202\,640\,600q^5 + 4\,252\,023\,300\,096q^6 + \ldots \end{aligned}$$

and the degrees d_i of characters of the Monster $\mathbb{M}$

$$\begin{aligned} &d_0 = 1, \; d_1 = 196\,883, \; d_2 = 21\,296\,876, \; d_3 = 842\,609\,326, \\ &d_4 = 18\,538\,750\,076, \; d_5 = 19\,360\,062\,527, \; d_6 = 293\,553\,734\,298, \; \ldots \end{aligned}$$

Further numerology
Then

$$\begin{aligned} c_1 &= d_0 + d_1, \\ c_2 &= d_0 + d_1 + d_2, \\ c_3 &= 2d_0 + 2d_1 + d_2 + d_3, \\ c_4 &= 2d_0 + 3d_1 + 2d_2 + d_3 + d_5, \\ c_5 &= 4d_0 + 5d_1 + 3d_2 + 2d_3 + d_4 + d_5 + d_6, \\ &\ldots \end{aligned} \tag{2}$$

Based on these relations, McKay & Thompson conjectured the existence of a ('natural') infinite dimensional representation of $\mathbb{M}$

$$V = \bigoplus_{n \geq -1} V_n,$$

s.t. $\dim V_n = c_n$. Then the Hauptmodul $J(z)$ is the **graded dimension** of V.

Numerology for Leech lattice

Also McKay gave a relation between the j-function and the classical Lie algebra E_8 and between J and the Leech lattice. The Leech lattice is special lattice in 24 dimensions.

$$\begin{aligned} c_1 &= 196\,560 + 324 \cdot 1 \\ c_2 &= 16\,773\,120 + 24 \cdot 196\,560 + 3200 \cdot 1, \\ c_3 &= 398\,034\,000 + 24 \cdot 16\,773\,120 + 324 \cdot 196\,560 + 25\,560 \cdot 1, \end{aligned}$$

where
196 560 is the number of vectors in Leech lattice whose (squared) norm equals 4,
16 773 120 is the number of norm 6 vectors and
398 034 000 the number of norm 8 vectors.
The same equalities hold for any 24-dimensional even self-dual lattices, apart from an extra term on the right hand sides. This term corresponds to norm 2 vectors; there are none of these in Leech lattice.

McKay – Thompson series

Thompson [Thompson, J. Finite groups and modular functions. Bull. London Math. Soc. 11: 347 – 351, 1979]: for $\forall g \in \mathbb{M}$ consider the series

$$T_g(z) = \sum_{n \in \mathbb{Z}} trace(g|V_n)q^n,$$

where $q = e^{2\pi iz}$ and V_n is the nth graded component of V. This graded trace is called the **McKay – Thompson series** for g, and generalizes the Hauptmodul: if $g = 1$ (identity element in $\mathbb{M}$) then $T_g(z) = J(z)$. Remarkable numerology concerning these graded traces is comprised in [Conway J.H. & Norton S.P. Monstrous Moonshine. Bull. London Math. Soc. 11: 308 – 339, 1979]

Thompson, Conway and Norton conjecture

All the series they were discovering (proceeding experimentally by first few coefficients) were normalized generators if genus 0 function fields arising from certain discrete subgroups of $PSL_2(\mathbb{R})$. They came to the **conjecture**:
there is a graded representation V of $\mathbb{M}$ with all the functions $T_g(z)$ have genus zero property.
Such graded module was discovered by Frenkel, Lepowsky and Meurman [Frenkel I., Lepowsky J. and Meurman A. A natural representation of the Fischer – Griess monster with the modular function J as a character. Proc. Natl. Acad. Sci. USA, 81: 3256 – 3260, 1984, Vertex Operator Algebra and the Monster. Acad. Press, NY, 1988], and was called the **monster vertex algebra**, or the **moonshine module** $V^\natural$.

More congruence groups

A subgroup $G \subset PSL_2(\mathbb{R})$ is **commensurable with** $PSL_2(\mathbb{Z})$ if both indices $[PSL_2(\mathbb{Z}) : (PSL_2(\mathbb{Z}) \cap G)]$ and $[G : (PSL_2(\mathbb{Z}) \cap G)]$ are finite. Consider the action $G \curvearrowright H$. Since G is commensurable with $PSL_2(\mathbb{Z})$ then H/G is a compact Riemann surface minus finite set of points. Then $H/G \subset \overline{H/G}$ where $\overline{H/G}$ is compact Riemann surface of genus g.
When $g = 0$ the field of automorphic functions $\overline{H/G} \to \mathbb{CP}^1$ is generated by just one element, and we can take this element J_G as the fixed isomorphism of Riemann surfaces, with leading coefficient 1 and constant term 0.
$J_G(z)$ is the **canonical isomorphism** or **normalized Hauptmodul** of G. J_G plays the same role for G as J plays for $PSL_2(\mathbb{Z})$.
Examples.

$$\begin{aligned} J_{\Gamma_0(2)} &= q^{-1} + 276q - 2018q^2 + 11202q^3 - 49152q^4 + 18402q^5 + \dots, \\ J_{\Gamma_0(13)} &= q^{-1} - q + 2q^2 + q^3 + 2q^4 - 2q^5 - 2q^7 - 2q^8 + q^9 + \dots, \\ J_{\Gamma_0(25)} &= q^{-1} - q + q^4 + q^6 - q^{11} - q^{14} + q^{21} + q^{24} - q^{26} + \dots. \end{aligned}$$

Moonshine Conjecture of Conway and Norton:

For each $g \in \mathbb{M}$ the McKay – Thompson series $T_g(z)$ is the normalized Hauptmodul $J_G : \overline{H/G} \to \mathbb{CP}^1$ for some subgroup $G \subset SL_2(\mathbb{R})$ commensurable with $PSL_2(\mathbb{Z})$.

A **Moonshine** for a finite group G is a *pair* $(\mathbb{G}, \phi)$ where

- $\phi : \mathbb{G} \to \mathfrak{F}$

$$\mathfrak{F} = \left\{ J_G : H \to \mathbb{C} \;\middle|\; \begin{array}{l} 1)\ J_G(z) \text{ modular w.r.t. discrete sbgr } G \leq SL_2(\mathbb{R}) \\ \exists N \in \mathbb{N} : \Gamma_0(N) \subseteq G \\ 2)\ genus(\overline{H/G}) = 0,\ \ \mathbb{C}(\overline{H/G}) = \mathbb{C}(J_G) \\ 3) \text{ in a neighborhood of } \infty \\ J_G(z) = \frac{1}{q} + \sum_{n=1}^{\infty} c_n q^n,\ \ q = e^{2\pi iz}, z \in H, c_n \in \mathbb{C} \end{array} \right\}$$

- if $\forall g \in \mathbb{G}$ $\phi_g(z) = \frac{1}{q} + \sum_{n=1}^{\infty} c_n(\sigma)q^n$, $q = e^{2\pi iz}$

then the map $\mathbb{G} \to \mathbb{C} : g \mapsto c_n(g)$ is a McKay– Thompson series of $\mathbb{G}$.

Outline of Borcherds' proof

"Monstrous moonshine and monstrous Lie superalgebras"Invent. Math. 102 (1992) 405 – 444

1. Construct a **vertex operator algebra** V, a graded algebra affording the moonshine representations of $\mathbb{M}$.
2. Construct a Lie algebra $\mathfrak{M}$ from V; this $\mathfrak{M}$ is a **generalized Kac – Moody Lie algebra**.
3. Construct a **denominator identity** for $\mathfrak{M}$ related to the coefficients of $J(q)$.
4. Construct **twisted denominator identities** related to the series $T_g(q)$.
5. Complete the proof.

Vertex Algebra

Let $\mathbb{F}$ be a field.

A **vertex algebra** is a k-vector space V with a collection of bilinear maps $\cdot_n : V \times V \to V : (u,v) \mapsto u \cdot_n v$ for all $m \in \mathbb{Z}$.

Maps $\cdot_m$ satisfy following axioms:

1. For $n \gg 0$ $u \cdot_n v = 0$. ($\exists n_0 > 0$ (depending on u, v s.t. $\forall n \geq n_0$ $u \cdot_n v = 0$).
2. $\exists 1 \in V$ (physics notation $|0\rangle$ – vacuum vector) s.t. $1 \cdot_{-1} v = v$, $1 \cdot_n v = 0$ $\forall n \neq -1$; $v \cdot_{-1} 1 = v$, $v \cdot_n 1 = 0$ $\forall n \geq 0$.
3. (Borcherds' identity=Jacobi identity for vertex algebras) $\forall u, v, w \in V$ $\forall m, n \in \mathbb{Z}$

$$\sum_{i \geq 0} \left(\begin{array}{c} m \\ i \end{array} \right) (u \cdot_{n+i} v) \cdot_{m+k-i} w =$$

$$\sum_{i \geq 0} (-1)^i \left(\begin{array}{c} n \\ i \end{array} \right) [u \cdot_{m+n-i} (v \cdot_{k+i} w) - (-1)^n v \cdot_{k+n-i} (u \cdot_{m+i} w)]$$

We restrict ourselves by $\mathbb{F} = \mathbb{R}$.

Vertex Operator

Denote $End\, V[[z, z^{-1}]] = \{\sum_{n \in \mathbb{Z}} \varphi_n z^{n-1} | \varphi_n \in End_k V \forall n \in \mathbb{Z}\}$.

A **vertex operator** $Y(u,z) : V \to V$ is defined by $Y(u,z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$ for $u_n \in End V$ given by n-th v.a. multiplication $v \mapsto u \cdot_n v$.

$$\delta(z - w) := z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n \in \mathbb{F}[[z, z^{-1}, w, w^{-1}]]$$

The v.a. axioms take a view

1. $Y(u,z)v$ has coefficient 0 at z^n for all n sufficiently small.
2. $\exists 1 \in V$ s.t. $\bullet$ $Y(1,z) = 1_V \in End(V)$; $\bullet$ $Y(v,z)1 \in V[[z]]$ $\forall v \in V$; $\bullet$ $\lim_{z \to 0} Y(v,z)1 = v$.
3. $\delta(z_1 - z_2) Y(u, z_1) Y(v, z_2) - \delta(z_2 - z_1) Y(v, z_2) Y(u, z_1) = \delta(z_1 - z_0) Y(Y(u, z_0)v, z_2)$ for $u, v \in V$.

Translation and Derivation

For a series $a(z) = \sum_n a_n z^n \in V[[z, z^{-1}]]$ denote $\partial a(z) = \sum_n n a_n z^{n-1}$.

Let $T : V \to V$ be the linear map s.t. $T(v) = v \cdot_{-2} 1$.

Theorem. There are following equivalent axioms for a v.a.:

1'. (translation covariance) $[T, Y(u,z)] = \partial Y(u,z)$ $\forall u \in V$;

2'. (vacuum) $Y(1,z) = 1_V$, $Y(u,z)1|_{z=0} = u$ $\forall u \in V$;

3'. (locality) $(z-w)^n Y(u,z) Y(v,w) = (z-w)^n Y(v,w) Y(u,z)$ for $n \gg 0$ (depending on u, v).

The application T repeatedly to the equation $Tv = v \cdot_{-2} 1$ and the identity $Tv_n = [t, v_n] + v_n T$ lead to $T(v_n 1) = (-n) v_{n-1} 1$; $v_{-n} 1 = \frac{1}{(n-1)!} T^{n-1} v$ $\forall n \geq 1$.

Hence $Y(u,z)1 = e^{z^T} u$ $\forall n \in V$.

Remark The bracket operation $[u, v] = u_0 v$ makes V/TV into a Lie algebra. Also if $a(z) = \sum_n a_n z^{-n-1} \in End\, V[[z, z^{-1}]]$ define

$$a(z)_+ = \sum_{n<0} a_n z^{-n-1} = a_{-1} + a_{-2} z + a_{-3} z^2 + \ldots,$$
$$a(z)_- = \sum_{n \geq 0} a_n z^{-n-1} = a_0 z^{-1} + a_1 z^{-2} + a_2 z^{-3} + \ldots.$$

Given $a(z), b(z)$, define a **normal ordered product** as

$$: a(z) b(z) := a(z)_+ b(z) + b(z) a(z)_-.$$

Borcherds' identity is equivalent to the following three identities

a) $[u_m, Y(v,z)] = \sum_{i \geq 0} \left(\begin{array}{c} m \\ i \end{array} \right) Y(u_i v, z) z^{m-i}$, $\forall u, v \in V$, $\forall m \in \mathbb{Z}$;

b) $: Y(u,z) Y(v,z) := Y(u_{-1} v, z)$ $\forall u, v \in V$;

c) $Y(Tu, z) = \partial Y(u,z)$ $u \in V$.

Conformal vector

An element ω of a v.a. V is a **conformal vector of central charge** c if it is an even vector satisfying
• $\omega_0 v = Tv\ \forall v \in V$, • $\omega_1\omega = 2\omega$, • $\omega_2\omega = 0$, • $\omega_3\omega = \frac{c}{2}1$, • $\omega_i\omega = 0\ \forall i \geq 4$,
• $V = \bigoplus_{n\in\mathbb{Z}} V_n$ where $V_n = \{v \in V | \omega_1 v = nv\}$.
In other words, ω is conformal vector if the corresponding vertex operator $Y(\omega, z)$ is a **Virasoro field with central charge** c, i.e. a formal series $L(z)$ satisfying

$$L(z)L(w) = \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}.$$

In particular, $1 \in V_0$ and $\omega \in V_2$.
When a v.a. V has a conformal vector, it admits an action of the Lie algebra called a **Virasoro algebra**.
A v.a. endowed with a conformal vector ω of central charge c is called a **vertex operator algebra** (or a **conformal vertex algebra**) **of rank** c.

Derivation Lie algebra

For $p(t) \in \mathbb{F}[t, t^{-1}]$ consider the derivation

$$T_{p(t)} = p(t)\partial : \mathbb{F}[t, t^{-1}] \to \mathbb{F}[t, t^{-1}].$$

The derivations of this form constitute a Lie algebra $\mathfrak{d}$ with natural bracket operation

$$[T_{p(t)}, T_{q(t)}] = T_{p(t)q'(t)-p'(t)q(t)}$$

for $p(t), q(t) \in \mathbb{F}[t, t^{-1}]$.
Choose a basis $d_n = -t^{n+1}\partial$, $n \in \mathbb{Z}$, in $\mathfrak{d}$.
Proposition. All the derivations of $\mathbb{F}[t, t^{-1}]$ form the Lie algebra $\mathfrak{d}$.
Any three generators d_n, d_0, d_{-n}, $n \in \mathbb{N}$ span a subalgebra isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ of traceless 2×2-matrices. Single out the subalgebra $\mathfrak{p} = \mathbb{F}d_1 + \mathbb{F}d_0 + \mathbb{F}d_{-1}$.

Virasoro algebra

Denote by $\mathfrak{v}$ the 1-dimensional central extension of $\mathfrak{d}$ with basis consisting of a central element c and elements L_n, $n \in \mathbb{Z}$, corresponding to the basis elements d_n of $\mathfrak{d}$. For $m, n \in \mathbb{Z}$ set

$$\begin{aligned}[][L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \\ &= (m-n)L_{m+n} + \frac{1}{2}\begin{pmatrix} m+1 \\ 3 \end{pmatrix}\delta_{m+n,0}c\end{aligned}$$

and $[L_m, c] = 0$, $\forall m \in \mathbb{Z}$. This central extension is trivial when restricted to the subalgebra $\mathfrak{p} \subset \mathfrak{d}$.
Proposition. The extension $\mathfrak{v}$ of the Lie algebra $\mathfrak{d}$ is the unique nontrivial 1-dimensional extension up to isomorphism.
The Lie algebra $\mathfrak{v}$ is called a **Virasoro algebra**.

Action on a Vertex Operator Algebra

Let V be a vertex operator algebra with conformal vector ω. Define the map $L_n : V \to V$ by $L_n = \omega_{n+1}$ for $n \in \mathbb{Z}$. By properties of ω

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{2}\begin{pmatrix} m+1 \\ 3 \end{pmatrix}\delta_{m+n,0}c1_V.$$

Thus, V is acted upon by the Virasoro algebra in which the central element c acts on V as $c1_V$, and c is the central charge of ω.

Return to Lie Algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. It has a decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$$

where

$$\mathfrak{n}^+ = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha,$$

and all $\mathfrak{g}_\alpha$ satisfy $\dim \mathfrak{g}_\alpha = 1$ and $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$. Here $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra, $\mathfrak{g}_\alpha$ are root spaces w.r.t. $\mathfrak{h}$.
Each of these root spaces gives rise to 1-dimensional representation α of $\mathfrak{h}$ defined by

$$[x, x_\alpha] = \alpha(x) \cdot x_\alpha$$

for all $x_\alpha \in \mathfrak{g}_\alpha$ and $x \in \mathfrak{h}$.

Root lattice. Cartan matrix

We use the standard notation $\Phi = \Phi^+ \cup \Phi^-$ for the set of roots of $\mathfrak{g}$.
Let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi$ be the subset of simple roots, $r = \dim \mathfrak{h}$.
Roots in Φ^+ are linear combinations of elements of Δ with non-negative integer coefficients; roots in Φ^- are linear combinations of elements of Δ with non-positive integer coefficients.
The free abelian group $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_r$ is called the **root lattice**.
The real vector space $Q \otimes_{\mathbb{Z}} \mathbb{R}$ inherits a natural structure of Euclidean space.

Let w_i be the reflection in the wall orthogonal to α_i. The isometry group of $Q \otimes_{\mathbb{Z}} \mathbb{R}$ is generated by $w_1, \ldots, w_r$ in the Weyl group W of $\mathfrak{g}$. This is a finite group permuting the elements of Φ. Each root is the image of some simple root under an element of W. $w_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$ for $A_{ij} \in \mathbb{Z}$.

The matrix $A = (A_{ij})$ is the **Cartan matrix** of $\mathfrak{g}$ w.r.t. Cartan subalgebra $\mathfrak{h}$.

Generalized Cartan matrix and Kac – Moody algebra

Properties of any Cartan matrix:
• $A_{ii} = 2\ \forall i = 1, \ldots, r$; • $A_{ij} \in \{0, -1, -2, -3\}$ if $i \neq j$; • $A_{ij} = 0$ iff $A_{ji} = 0$; • $A_{ij} = \{-2, -3\}$ implies $A_{ji} = -1$.
The Lie algebra $\mathfrak{g}$ can be defined by generators and relations depending only on the Cartan matrix A.
The Cartan subalgebra $\mathfrak{h}$ has a basis $h_1, \ldots, h_r$ s.t.$\alpha_j(h_i) = A_{ij}$.

The **generalized Cartan matrix** (**GCM**) is any matrix satisfying the conditions
• $A_{ij} \in \mathbb{Z}$; • $A_{ii} = 2\ \forall i$; • $A_{ij} \leq 0$, if $i \neq j$; • $A_{ij} = 0$ iff $A_{ji} = 0$.
A Lie algebra is defined by generators and relations depending on the GCM A, as in the case of finite-dimensional simple Lie algebras. This Lie algebra is extended by outer derivations to ensure the simple roots to be linearly independent even though A occur to be singular.

The resulting Lie algebra is called the **Kac – Moody algebra** given by GCM A.

Main differences from finite dimensional case are that:
1. The Lie algebra $\mathfrak{g}$ can be infinite dimensional.
2. The root spaces $\mathfrak{g}_\alpha$ can have dimension greater then 1.
3. The Weyl group W can be infinite.
4. There can be both real and imaginary roots.

The root α is **real** if $\langle \alpha, \alpha \rangle > 0$, the root α is **imaginary** if $\langle \alpha, \alpha \rangle \leq 0$. All simple roots of a Kac – Moody algebra are real, any real root can be obtained from a simple root by the action of appropriate element of the Weyl group.

The GCM A is **symmetrizable** if $A = DB$ where B is symmetric and D is nonsingular diagonal matrix. We restrict to Kac – Moody algebras with symmetrizable GCM.
We use the abbreviation **SKMA** for Symmetrized Kac – Moody Algebra.

Binary Code

Let Ω be a finite set, $|\Omega| = n$. The power set $2^\Omega = \{S : S \subseteq \Omega\}$ can be viewed as $\mathbb{F}_2$-vector space under the operation $+$ of symmetric difference.
Binary linear code is a $\mathbb{F}_2$-subspace of 2^Ω.
The cardinality $|C|$ of an element $C \in \mathcal{C}$ is the **weight** of C.
A code $\mathcal{C}$ is **of type I** if $n \in 2\mathbb{Z}$, $\Omega \in \mathcal{C}$ and $\forall C \in \mathcal{C}\ |C| \in 2\mathbb{Z}$.
A code $\mathcal{C}$ is **of type II** if $n \in 4\mathbb{Z}$, $\Omega \in \mathcal{C}$ and $\forall C \in \mathcal{C}|C| \in 4\mathbb{Z}$.
The **dual code** $\mathcal{C}^\circ$ for $\mathcal{C}$ is

$$\mathcal{C}^\circ = \{S \subset \Omega | |S \cap C| \in 2\mathbb{Z}\ \forall C \in \mathcal{C}\}$$

Thus $\mathcal{C}^\circ$ is the annihilator of $\mathcal{C}$ in 2^Ω w.r.t. the natural nonsingular symmetric bilinear form $(S_1, S_2) \mapsto |S_1 \cap S_2| + 2\mathbb{Z}$ on 2^Ω. Hence $\dim_{\mathbb{F}_2}\mathcal{C}^\circ = n - \dim_{\mathbb{F}_2}\mathcal{C}$.
$\mathcal{C}$ is **self-dual** if $\mathcal{C}^\circ = \mathcal{C}$. In this case n is even and $\dim_{\mathbb{F}_2}\mathcal{C} = \frac{n}{2}$.

Consider the **even subspace** $\mathcal{E}(\Omega) = \{S \subseteq \Omega : |S| \in 2\mathbb{Z}\}$. The map

$$q : \mathcal{E}(\Omega) \to \mathbb{Z}_2, \quad S \mapsto \frac{|S|}{2} + 2\mathbb{Z}$$

is a quadratic form associated with the bilinear form $(S_1, S_2) \mapsto |S_1 \cap S_2| + 2\mathbb{Z}$. In case when n even $\mathbb{F}_2\Omega$ is the radical of q. A subspace of a space with a quadratic form is called **totally singular** if the quadratic form vanishes on it.
The **weight distribution** of the code $\mathcal{C}$ is

$$w(\mathcal{C}) = \sum_{C \in \mathcal{C}} q^{|C|} \in \mathbb{Z}[q].$$

A **Hamming code** is a self-dual code of type II on an 8-element set Ω.
It can be proven that its weight distribution is $1 + 14q^2 + q^8$.
Proposition. The Hamming code is unique up to isomorphism.

Golay code

A **(binary) Golay code** is a self-dual code $\mathcal{C}$ of type II s.t. $\mathcal{C}$ has no elements of weight 4, on a 24-element set.
The Golay code is unique up to isomorphism. Its weight distribution is

$$1 + 759q^8 + 2\,576q^{12} + 759q^{16} + q^{24}.$$

The 759 elements of the Golay code of weight 8 are called **octads**.
Theorem. The Golay code is generated by octads.
The authomorphism group $Aut\,\mathcal{C}$ of the Golay code is the Mathieu group M_{24}. This is nonabelian simple group.

Lattices

The **lattice of rank** n is a rank n free abelian group L provided with a rational-valued symmetric $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle : L\times L \to \mathbb{Q}$.
It is easy to prove that $\langle\cdot,\cdot\rangle : L \times L \to \frac{1}{r}\mathbb{Z} \subset \mathbb{Q}$ for some $r \in \mathbb{N}$.
A lattice isomorphism is called **isometry**.
L is **non-degenerate** if $\langle\alpha, L\rangle = 0$ implies $\alpha = 0$.
Given a lattice L; it can be canonically embedded in the n-dimensional $\mathbb{Q}$-vector space $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbb{Z}$-bilinear form extends to $\mathbb{Q}$-bilinear form $\langle\cdot,\cdot\rangle : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \to \mathbb{Q}$.
L is **non-degenerate** if $\mathbb{Q}$-form is non-degenerate, i.e. if $\det(\langle\alpha_i, \alpha_j\rangle)_{ij} \neq 0$ for $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$.
L is **even** if $\langle\alpha, \alpha\rangle \in 2\mathbb{Z}\ \forall\alpha \in Z$.
L is **integral** if $\langle\alpha, \beta\rangle \in \mathbb{Z}\ \forall\alpha, \beta \in L$.
Even lattice is integral.
L is **positive definite** if $\langle\alpha, \alpha\rangle > 0\ \forall\alpha \in L - \{0\}$.

Dualization

The **dual** of L is $L^\circ = \{\alpha \in L_{\mathbb{Q}} | \langle\alpha, L\rangle \subseteq \mathbb{Z}\}$.
L° is a lattice iff L is non-degenerate.
In this case is the **dual basis** $\alpha^1, \ldots, \alpha^n$ of a given basis $\alpha_1, \ldots, \alpha_n$, is defined by $\langle\alpha^i, \alpha_j\rangle = \delta_{ij}\ \forall i, j = 1, \ldots, n$.
L is integral iff $L \subseteq L^\circ$.
The lattice L is **self-dual** if $L = L^\circ$.
It is **unimodular** if $|\det(\langle\alpha_i, \alpha_j\rangle)_{ij}| = 1$.
L is self-dual iff it is integral and unimodular.
A self-dual code $\mathcal{C}$ if type II on a set Ω gives rise to a even unimodular lattice.
Let $\mathfrak{h} = \bigoplus_{k\in\Omega} \mathbb{F}\alpha_k$ be a vector space with a basis $\{\alpha_k | k \in \Omega\}$. Define a symmetric bilinear form

$$\langle\cdot,\cdot\rangle : \mathfrak{h} \times \mathfrak{h} \to \mathbb{F} : \quad (\alpha_k, \alpha_\ell) \mapsto \delta_{k,\ell}$$

for $k, \ell \in \Omega$.

For $S \subseteq \Omega$ set $\alpha_S = \sum_{k\in S} \alpha_k$ and define $Q = \bigoplus_{k\in\Omega} \mathbb{Z}\alpha_k$. For a code $\mathcal{C}$ on Ω define the positive definite lattice

$$L_0 = \{\sum_{k\in\Omega} m_k\alpha_k | m_k \in \frac{1}{2}\mathbb{Z}, \{k | m_k \in \mathbb{Z} + \frac{1}{2}\} \in \mathcal{C}\}.$$

Then L_0 is even iff $|C| \in 4\mathbb{Z}\ \forall C \in \mathcal{C}$. The dual lattice L_0° of L_0 is based on the dual code $\mathcal{C}^\circ$:

$$L_0^\circ = \{\sum_{k\in\Omega} | m_k \in \frac{1}{2}\mathbb{Z}, \{k | m_k \in \mathbb{Z} + \frac{1}{2}\} \in \mathcal{C}^\circ\}.$$

Proposition. A code $\mathcal{C}$ is self-dual of type II iff the corresponding lattice L_0 is even self-dual, or equivalently, even unimodular.
Define

$$\begin{aligned} L_0' &= \sum_{C\in\mathcal{C}} \mathbb{Z}\frac{1}{2}\alpha_C + \sum_{k\in\Omega} \mathbb{Z}(\frac{1}{4}\alpha_\Omega - \alpha_k) \\ &= \sum_{C\in\mathcal{C}} \mathbb{Z}\frac{1}{2}\alpha_C + \sum_{k,\ell\in\Omega} \mathbb{Z}(\alpha_k - \alpha_\ell) + \mathbb{Z}(\frac{1}{4}\alpha_\Omega - \alpha_{k_0}), \end{aligned}$$

where $k_0 \in \Omega$ is a fixed element. Since the lattice

$$L_0 \cap L_0' = \sum_{C\in\mathcal{C}} \mathbb{Z}\frac{1}{2}\alpha_C + \sum_{k,\ell\in\Omega} \mathbb{Z}(\alpha_k - \alpha_\ell)$$

has index 2 in both L_0 and L_0' then L_0' is unimodular iff L_0 is.
Necessary and sufficient condition for L_0' to be even is that $n = |\Omega| \in 8(2\mathbb{Z} + 1)$.
Proposition. If $n \in 8(2\mathbb{Z} + 1)$ and the code $\mathcal{C}$ is self-dual of type II then the corresponding lattice L_0' is even unimodular.

The **Leech lattice** is the even inimodular lattice $\Lambda = L_0'$ for the case $n = 24$ and $\mathcal{C}$ the Golay code.
For a lattice L and for $m \in \mathbb{Q}$ define

$$L_m = \{\alpha \in L | \langle \alpha, \alpha \rangle = m\}.$$

The Leech lattice has $\Lambda_2 = \emptyset$.
Proposition. The Leech lattice is the unique even unimodular lattice of rank 24 having no elements of norm 2.
Remark. Altogether, there are 24 even unimodular lattices of rank 24, up to isometry, called the **Niemeier lattices**.
The group of isometries of the Leech lattice is called the **Conway group** Co_0:

$$Co_0 = Aut(\Lambda, \langle \cdot, \cdot \rangle) = \{g \in Aut\,\Lambda : \langle g\alpha, g\beta \rangle = \langle \alpha, \beta \rangle \forall \alpha, \beta \in \Lambda\}.$$

Its quotient by the central subgroup $\{\pm 1\}$ is called the **Conway group** Co_1.

Shortest elements in Λ
For $S \subseteq \Omega$ let ϵ_S be the involution of $\mathfrak{h}$ s.t.

$$\epsilon_S : \alpha_k \mapsto \begin{cases} -\alpha_k & \text{if } k \in S \\ \alpha_k & \text{if } k \notin S \end{cases} \quad \text{for } k \in \Omega.$$

$\Lambda_4 = \Lambda_4^1 \sqcup \Lambda_4^2 \sqcup \Lambda_4^3$ where

$$\begin{aligned} \Lambda_4^1 &= \{\frac{1}{2}\epsilon_S \alpha_C : C \in \mathcal{C}, |C| = 8, S \subseteq C, |S| \in 2\mathbb{Z}\}; \\ \Lambda_4^2 &= \{\pm\alpha_k \pm \alpha_\ell : k, \ell \in \Omega, k \neq \ell\}; \\ \Lambda_4^3 &= \{\epsilon_C(\frac{1}{4}\alpha_\Omega - \alpha_k) : C \in \mathcal{C}, k \in \Omega\}. \end{aligned}$$

For cardinalities one has

$$|\Lambda_4| = |\Lambda_4^1| + |\Lambda_4^2| + |\Lambda_4^3| = 759 \cdot 2^7 + \begin{pmatrix} 24 \\ 2 \end{pmatrix} \cdot 2^2 + 24 \cdot 2^{12} = 196\,560.$$

Using the Leech lattice we form the **untwisted space** $V_\Lambda = S(\widetilde{\mathfrak{h}}^-) \otimes_{\mathbb{C}} \mathbb{C}[\Lambda]$ where

$$\widetilde{\mathfrak{h}}^- = \bigoplus_{n<0} (\mathfrak{h} \otimes t^n)$$

for $\mathfrak{h} = \Lambda \otimes_{\mathbb{Z}} t^n$, $S(\cdot)$ denotes formation of a symmetric algebra.
Let $\widehat{L}$ be a central extension of the lattice L by a finite cyclic group $< \kappa >= \{k | k^s = 1\}$ of order s, and denote $c_0 : L \times L \to \mathbb{Z}_s$ the associated commutator map s.t. $aba^{-1}b^{-1} = k^{c_0(\overline{a}, \overline{b})}$ for $a, b \in \widehat{L}$. We make choices: fix the central extension

$$1 \to < \kappa > \to \widehat{\Lambda} \to \Lambda \to 1$$

where $s = 2$ and where the commutator map is the alternating $\mathbb{Z}$-bilinear map $c_0(\alpha, \beta) = \langle \alpha, \beta \rangle mod 2\mathbb{Z}$ for $\alpha, \beta \in \Lambda$. Then $\mathbb{C}[\Lambda] = \mathbb{C}[\widehat{\Lambda}]/(\kappa + 1)\mathbb{C}[\Lambda]$ and $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ where $c : \Lambda \times \Lambda \to \mathbb{C}^\times : (\alpha, \beta) \mapsto \xi^{c_0(\alpha, \beta)}$. Here ξ is 2nd primitive root of unity $\xi = -1$. Then κ acts on V_Λ as multiplication by -1 and $ab = (-1)^{\langle \overline{a}, \overline{b} \rangle} ba$ for $a, b \in \widehat{\Lambda}$. The automorphisms of $\widehat{\Lambda}$ which induce the involution -1 on Λ are involutions and are parametrized by quadratic forms on $\Lambda/2\Lambda$ with associated form induced by c_0. We fix the involution θ_0 determined by the quadratic form $q_1 : \Lambda/2\Lambda \to \mathbb{Z}_2 : \alpha + 2\Lambda \mapsto \frac{1}{2}\langle \alpha, \alpha \rangle + 2\mathbb{Z}$. Then there is an involution $\theta_0 : \widehat{\Lambda} \to \widehat{\Lambda} : a \mapsto a^{-1}\kappa^{\langle \overline{a}, \overline{a} \rangle / 2}$. It has properties $\theta_0(a^2) = a^{-2}$ and for $\overline{a} \in \Lambda_4$ $\theta_0(a) = a^{-1}$.
It is possible to make V_Λ into a vertex algebra.
For all $\alpha \in \Lambda$

$$Y(1 \otimes e^\alpha, z) = \exp\left(\sum_{n<0} -\frac{\alpha(n)}{n} z^{-n}\right) \exp\left(\sum_{n>0} -\frac{\alpha(n)}{n} z^{-n}\right) e_\alpha z^\alpha,$$

where elements $\alpha(n), e_\alpha, z^\alpha \in End V_\Lambda$ are defined as follows.
$\alpha(n)$ is defined as follows. $\alpha(0) \in End \mathbb{C}[\Lambda]$ is given by $\alpha(0)e^\gamma = \gamma(\alpha)e^\gamma$, and $\alpha(n) \in End S(\widetilde{\mathfrak{h}}^-)$ satisfy
If $n < 0$ then $\alpha(n)$ is multiplication by $\alpha \otimes t^n$.
If $n > 0$ then $\alpha(n)$ is the derivation of $S(\widetilde{\mathfrak{h}}^-)$ determined by $\alpha(n)(x \otimes t^{-n}) = n(x, \alpha)$ for $x \in \mathfrak{h}$, $\alpha(n)(x \otimes t^{-m}) = 0$ if $m \neq n$.
$z^\alpha \in End V_\Lambda$ is given by $1 \otimes z^\alpha$ for $z^\alpha \in End \mathbb{C}[\Lambda]$, and $z^\alpha e^\gamma = z^{\langle \alpha, \gamma \rangle} e^\gamma$. Also $e_\alpha \in End V_\Lambda$ is given by $1 \otimes e_\alpha$ for $e_\alpha \in End \mathbb{C}[\Lambda]$, end $e_\alpha e^\gamma = \epsilon(\alpha, \gamma) e^{\alpha + \gamma}$, where $\epsilon : \Lambda \times \Lambda \to \{\pm 1\}$ is 2-cocycle.
More generally, for

$$v = (h_1 \otimes t^{-n_1}) \cdots (h_k \otimes t^{-n_k}) \otimes e^\alpha \in V_\Lambda$$

one has

$$Y(v, z) =: \frac{1}{(n_1 - 1)!} \left(\frac{d}{dz}\right)^{n_1 - 1} h_1(z) \cdots \frac{1}{(n_k - 1)!} \left(\frac{d}{dz}\right)^{n_k - 1} h_k(z) Y(1 \otimes e^\alpha, z) :$$

where the normal ordered product of more then two factors is defined inductively as follows

$$: a_1(z)a_2(z)\dots a_k(z) :=: a_1(z)(: a_2(z)\dots a_k(z) :) : .$$

For $h \in \mathfrak{h}$ and $m > 0$

$$Y((h \otimes t^{-m}) \otimes 1, z) = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} h(z)$$

where $h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}$ and $h(n) \in End V_\Lambda$ is defined as follows. $h(0) \in End\mathbb{C}[\Lambda]$ is given by $h(0)e^\gamma = \gamma(h)e^\gamma$, and $h(n) \in End S(\widetilde{\mathfrak{h}}^-)$ satisfy
If $n < 0$ then $h(n)$ is multiplication by $h \otimes t^n$.
If $n > 0$ then $h(n)$ is the derivation of $S(\widetilde{\mathfrak{h}}^-)$ determined by $h(n)(x \otimes t^{-n}) = n(x, h)$ for $x \in \mathfrak{h}$, $h(n)(x \otimes t^{-m}) = 0$ if $m \neq n$.

Also V_Λ has a conformal vector ω given by

$$\omega = \left(\frac{1}{2}\sum_{i=1}^{r}(h_i' \otimes t^{-1})(h_i \otimes t^{-1})\right) \otimes 1 \in V_\Lambda$$

where $h_1, \dots, h_r$ is a basis for $\mathfrak{h}$, and $h_1', \dots, h_r'$ is the dual basis w.r.t. $\langle\cdot,\cdot\rangle$. The element ω is independent of the choice of basis.

The maps $L_i : V_\Lambda \to V_\Lambda$ defined by $L_i = \omega_{i+1}$ satisfy

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{1}{2}\binom{i+1}{3}\dim\mathfrak{h}\,\delta_{i+j,0}1_{V_\Lambda}.$$

Thus V_Λ is a module for the Virasoro algebra where the central element c acts as multiplication by $\dim\mathfrak{h} = \operatorname{rank}\Lambda = 24$. Also form a **twisted space**

$$V_\Lambda^T = S(\widetilde{\mathfrak{h}}^-) \otimes_{\mathbb{Z}+\frac{1}{2}} T$$

where T is any Λ-module s.t. $\kappa \cdot v = \xi v\ \forall v \in T$.
Preserving the choices done before we set

$$K = \{\theta_0(a)a^{-1} : a \in \widehat{\Lambda}\} = \{a^2\kappa^{\langle\overline{a},\overline{a}\rangle/2} : a \in \widehat{\Lambda}\}$$

which is a central subgroup of $\widehat{\Lambda}$ s.t. $K = 2\Lambda$. Then $\widehat{\Lambda}/K$ is a finite group which is a central extension $1 \to \langle\kappa\rangle \to \widehat{\Lambda}/K \to \Lambda/2\Lambda \to 1$ with the commutator map $aba^{-1}b^{-1} = \kappa^{c_0(a,b)}$ for $a, b \in \widehat{L}$, $c_0 : L \times L \to \mathbb{Z}_s$ induced by $c_0(\alpha,\beta) = \langle\alpha,\beta\rangle + 2\mathbb{Z}$ as before, and with squaring map the quadratic form q_1. Since Λ is unimodular, q_1 is nonsingular, and $\widehat{\Lambda}/K$ is an extraspecial 2-group with $|\widehat{\Lambda}/K| = 2^{25}$. In fact, in the twisted space V_Λ^T we take T to be the canonical $\widehat{\Lambda}$-module $V_\Lambda = S(\widetilde{\mathfrak{h}}^-) \otimes_{\mathbb{Z}} \mathbb{C}[\Lambda]$. Of course for $a \in \widehat{\Lambda}$ $\theta_0(a) = a$ as operators on T.

Define a **Moonshine Module** acted upon by the Monster group.
Actions of θ_0:
on V_Λ as $\theta_0 : x \otimes i(a) \mapsto \theta_0(x) \otimes i(\theta_0(a))$ for $x \in S(\widetilde{\mathfrak{h}}^-)$ and $a \in \widehat{\Lambda}$;
on V_Λ^T as $x \otimes \tau \mapsto \theta_0(x) \otimes (-\tau) = -\theta_0(a) \otimes \tau$ for $x \in S(\widetilde{\mathfrak{h}}^-)$ and $\tau \in T$.

Let $V_\Lambda^{\theta_0}$ and $(V_\Lambda^T)^{\theta_0}$ be the subspaces of V_Λ and (V_Λ^T) of θ_0-invariant elements.
We know that for $v \in V_\Lambda^{\theta_0}$ the component operators of both untwisted and twisted vertex operators $Y(v, z)$ preserve the respective fixed spaces $V_\Lambda^{\theta_0}$ and $(V_\Lambda^T)^{\theta_0}$.

The **Moonshine Module** is the space $V^\natural = V_\Lambda^{\theta_0} \oplus (V_\Lambda^T)^{\theta_0}$.

Vertex operator on a Moonshine Module
For $v \in V_\Lambda$ form the vertex operator

$$Y(v, z) = Y_{\mathbb{Z}}(v, z) \oplus Y_{\mathbb{Z}+\frac{1}{2}}(v, z)$$

acting on the larger space

$$W_\Lambda = V_\Lambda \oplus V_\Lambda^T$$

so that for $v \in V$ and for $x_v(n) \in W_\Lambda$

$$v_n V^\natural \subseteq V^\natural, \quad x_v(n)V^\natural \subseteq V^\natural,$$

if $v \in V_\Lambda^{\theta_0}$.
$V^\natural$ can be given a structure of a vertex operator algebra.
There is a conformal vector ω of central charge 24. Thus the linear maps $L_i : V^\natural \to V^\natural$ satisfy

$$[L_i, L_j] = (i-j)L_{i+j} + 12\binom{i+1}{3}\delta_{i+j,0}1_{V^\natural}$$

and thus give a representation of the Virasoro algebra in which the image of the central element c equals $24 \cdot 1_{V^\natural}$.

Structure of $V^\natural$

• Integral grading $V^\natural = \bigoplus_{n\in\mathbb{Z}} V_n^\natural$ with $V_n^\natural = 0$ for $n < -1$.

In particular, $V_1^\natural = \mathfrak{f} \oplus \mathfrak{p}$, where

$$\mathfrak{f} = S^2(\mathfrak{h}) \oplus \sum_{\alpha\in\widehat{\Lambda}_4} \mathbb{F}x_\alpha^+, \quad \mathfrak{p} = \mathfrak{h} \oplus T.$$

Define a **Griess module** as a space $\mathcal{B} = V_1^\natural = \mathfrak{f} \oplus \mathfrak{p}$.

Dimensions of first three direct summands are

$$\dim V_{-1}^\natural = 1, \quad \dim V_0^\natural = 0, \quad \dim V_1^\natural = \dim \mathcal{B} = 196\,884.$$

In fact we have

$$\sum_{n\in\mathbb{Z}} (\dim V_n) q^n = J(z) = q^{-1} + 0 + 196\,884q + 21\,493\,760q^2 + \ldots,$$

where $q = e^{2\pi iz}$, $z \in H$.

$\mathcal{B}$ can be given the structure of a commutative associative algebra with identity and with a nonsingular symmetric associative form in the following way: the product $\times$ and the form $\langle\cdot,\cdot\rangle$ on $\mathcal{B}$ extend those on $\mathfrak{f}$: $u\times v = v\times u = u_1\cdot v$, $\langle u, v\rangle = \langle v, u\rangle = u_3\cdot v = 0$. The identity element $\frac{1}{2}\omega$ in $\mathfrak{f}$ is also identity on $\mathcal{B}$. Define on $\mathfrak{p} = \mathfrak{h}\otimes T$: $\langle h_1\otimes\tau_1, h_2\otimes\tau_2\rangle = \frac{1}{2}\langle h_1, h_2\rangle\langle\tau_1,\tau_2\rangle$ for $h_i \in \mathfrak{h}$, $\tau_i \in T$. The product $\times$ on $\mathfrak{p}$ is defined so that $\mathfrak{p}\times\mathfrak{p} \subseteq \mathfrak{f}$ and uniquely determined by nonsingularity of the form on $\mathfrak{f}$ and the associativity

$$\langle u, v\times w\rangle = \langle u\times v, w\rangle.$$

The resulting nonassociative algebra is called a **Griess algebra**.
Its automorphism group is a monster group $\mathbb{M}$.

The Monster group $\mathbb{M}$ acts as a group of automorphisms of $V^\natural$.
The subgroup of $\mathbb{M}$ preserving subspaces $V_\Lambda^{\theta_0}$ and $V_{\Lambda^T}^{\theta_0}$, is the centralizer of the involution in $\mathbb{M}$. It is an extension of the extraspecial group $\widehat{\Lambda}/K$ of order 2^{25} by Conway's sporadic group Co_1 which is related to the Leech lattice Λ.
A crucial part of the Frenkel – Lepowsky – Meurman construction is to find an involution in the Monster $\mathbb{M}$ which acts on $V^\natural$ but does not preserve the subspaces $V_\Lambda^{\theta_0}$ and $(V_\Lambda^T)^{\theta_0}$.

The conformal vector $\omega \in V^\natural$ lies in $\mathbb{M}$-invariant 1-dim subspace $\mathbb{C}\omega \subseteq V_1^\natural$. The complementary submodule of $\mathbb{C}\omega$ in $V_1^\natural$ gives the smallest nontrivial representation of $\mathbb{M}$ of degree $196\,883$. This is an explanation of McKay's observation.

The Monster vertex algebra $V^\natural$ is a graded module whose graded components have dimensions given by the coefficients of the normalized j-function $J(z)$. The Monster acts on each graded component.

On $V^\natural$, the Jacobi identity takes a simple form for vertex operators parametrized by $V_\Lambda^{\theta_0}$.

Theorem. For $v \in V_\Lambda^{\theta_0}$

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \text{ on } V^\natural,$$

that is, $Y(v,z)$ involves only integral powers of z. For $u, v \in V_\Lambda^{\theta_0}$ one has

$$\begin{aligned}
&[Y(u,z_1) \times_{z_0} Y(v,z_2)] \\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1)) \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(u,z_0)v,z_2)
\end{aligned}$$

on $V^\natural$. In particular, $V^\natural$ is a vertex operator algebra of ventral charge 24 and $(V_\Lambda^T)^{\theta_0}$ is a $V_\Lambda^{\theta_0}$-module.

Borcherds Lie Algebra

A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called a **Borcherds algebra** if
(i) $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ has a $\mathbb{Z}$-grading s.t. $\dim \mathfrak{g}_i$ is finite $\forall i \neq 0$.
(ii) There exists a linear map $\omega : \mathfrak{g} \to \mathfrak{g}$ s.t.
• $\omega^2 = 1$ (the identity map on $\mathfrak{g}$); • $\omega(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ $\forall i \in \mathbb{Z}$; • $\omega = -1$ on $\mathfrak{g}_0$.
(iii) $\mathfrak{g}$ has an invariant bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$, s.t.
• $\langle x, y\rangle = 0$ if $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$ and $i + j \neq 0$; • $\langle\omega x, \omega y\rangle = \langle x, y\rangle$ $\forall x, y \in \mathfrak{g}$; • $-\langle x, \omega x\rangle > 0$ if $x \in \mathfrak{g}_i$, $i \neq 0$, $x \neq 0$.
These axioms imply that $\mathfrak{g}_0$ is abelian and that the scalar product

$$\langle\cdot,\cdot\rangle_0 : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R} : \quad \langle x, y\rangle_0 = -\langle x, \omega y\rangle,$$

is positive definite on $\mathfrak{g}_i$ $\forall i \neq 0$.
This $\langle\cdot,\cdot\rangle_0$ is called the **contravariant bilinear form** on $\mathfrak{g}$.

Examples of Borcherds algebras

I. **Universal Borcherds algebra (UBA).** Let $a = (a_{ij})$, $i, j \in I$, be a symmetric matrix with $a_{ij} \in \mathbb{R}$. The set I is assumed to be either finite or countably infinite. Assume a to satisfy the conditions:
• $a_{ij} \leq 0$ if $i \neq j$, • if $a_{ii} > 0$ then $a\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}\ \forall j \in I$.

There is a Borcherds algebra $\mathfrak{g}$ associated to the matrix a. It is generated by elements e_i, f_j, h_{ij}, for $i, j \in I$, subject to the relations:
• $[e_i, f_j] = h_{ij}$, • $[h_{ij}, h_{kl}] = 0$, • $[h_{ij}, e_k] = \delta_{ij} a_{ik} e_k$, • $[h_{ij}, f_k] = -\delta_{ij} a_{ik} f_k$, • if $a_{ii} > 0$ and $i \neq j$ then $(\mathrm{ad} e_i)^n e_j = 0$, $(\mathrm{ad} f_i)^n f_j = 0$, where $n = 1 - 2\frac{a_{ij}}{a_{ii}}$, • if $a_{ii} \leq 0$, $a_{jj} \leq 0$ and $a_{ij} = 0$, then $[e_i, e_j] = 0$, $[f_i, f_j] = 0$.

This Lie algebra $\mathfrak{g}$ can be graded by the condition $\deg e_i = n_i$, $\deg f_i = -n_i$ for some $n_i \in \mathbb{Z}_+$.

There is an involution $\omega : \mathfrak{g} \to \mathfrak{g} : \omega(e_i) = -f_i$, $\omega(f_i) = -e_i$.

There is also an invariant bilinear form uniquely determined by $\langle e_i, f_i \rangle = 1$ for all $i \in I$. We write $h_i := h_{ii}$. Then, $\langle e_i, f_i \rangle = h_i$ and

$$\langle h_i, h_j \rangle = \langle [e_i, f_i], h_j \rangle = \langle e_i, [f_i, h_j] \rangle = \langle e_i, a_{ii} f_i \rangle = a_{ij}$$

for all $i \neq j$. Thus, $\langle h_i, h_j \rangle = a_{ij}$ for all $i.j \in I$.

We see that the Lie algebra $\mathfrak{g}$ satisfies the axioms of the Borcherds algebra. It is called the **universal Borcherds algebra** associated with the matrix (a_{ij}).

II. **UBA from SKMA.** Any symmetrizable Kac – Moody algebra over $\mathbb{R}$ gives rise to a universal Borcherds algebra. Let $\mathfrak{g}$ be the Kac – Moody algebra over $\mathbb{R}$ with symmetrizable GCM $A = (A_{ij})$. Thus there exists a diagonal matrix $D = diag\{d_1, \ldots, d_n\}$, $d_i \in \mathbb{Z}_+$, s.t. DA is symmetric. Let $a = (a_{ij})$ be given by $a_{ij} = d_i A_{ij}/2$ for all i, j. Then we have $a_{ij} = a_{ji}$ and $a_{ii} = d_i$. Thus $a_{ij} \leq 0$ if $i \neq j$ and $a_{ii} \in \mathbb{Z}_+$. Also $2a_{ij}/a_{ii} = A_{ij}$. Thus the symmetric matrix (a_{ij}) satisfies the conditions needed to construct a Borcherds algebra. The universal Borcherds algebra with symmetric matrix (a_{ij}) coincides with the subalgebra of the Kac – Moody algebra $\mathfrak{g}$ obtained by generators and relations prior to the adjunction of the commutative algebra of outer derivations.

The **difference between SKMA and UBA** is that in a Borcherds algebra
1. The index set I may be countably infinite rather then finite;
2. The a_{ii}'s may not be possible and need not lie in $\mathbb{Z}$;
3. $2\frac{a_{ij}}{a_{ii}}$ is only assumed to lie in $\mathbb{Z}$ when $a_{ii} > 0$.

The center of a UBA $\mathfrak{g}$ lies in in the abelian subalgebra generated by the elements h_{ij} and contains all h_{ij} with $i \neq j$. It can be seen that $h_{ij} = 0$ unless the i-th and j-th columns are equal. If we factor out the ideal $J \subset \mathfrak{g}$ which lies in the center, then $\mathfrak{g}/J$ retains the structure of a Borcherds algebra. If we adjoin to $\mathfrak{g}/J$ an abelian Lie algebra $\mathfrak{a}$ of outer derivations we obtain a Lie algebra $\mathfrak{g}^* = (\mathfrak{g}/J) \cdot \mathfrak{a}$ where $\mathfrak{a} \subseteq (\mathfrak{g}^*)_0$ and $[e_i, x] \in \mathbb{R}e_i$, $[f_i, x] \in \mathbb{R}f_i$ for all $x \in \mathfrak{a}$.
$\mathfrak{g}^*$ retains the structure of a Borcherds algebra.

Why Universal?

For any Borcherds algebra $\mathfrak{g}$ there are
✓ a unique UBA $\mathfrak{g}_U$; ✓ a (not necessarily unique!) homomorphism $f : \mathfrak{g}_U \to \mathfrak{g}$
s.t.
• $\ker f$ lies in the center of $\mathfrak{g}_U$. • $\mathrm{im}\, f$ is an ideal of $\mathfrak{g}$. • $\mathfrak{g}$ is a semidirect product of $\mathrm{im}\, f$ with a commutative Lie algebra of outer derivations lying in the 0-graded component of $\mathfrak{g}$ and preserving all subspaces $\mathbb{R}e_i$ and $\mathbb{R}f_i$.

The homomorphism f preserves grading, involution and bilinear form.

Roots of universal Borcherds algebra

Let $\mathfrak{g}$ be the universal Borcherds algebra.
Recall that the root lattice Q of $\mathfrak{f}$ is a free abelian group with basis r_i, for $i \in J$, with symmetric bilinear form

$$Q \times Q \to \mathbb{R} : \quad (r_i, r_j) \mapsto \langle r_i, r_j \rangle = a_{ij}.$$

The basis elements r_i are called **simple roots**. We have a grading

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$$

determined by $e_i \in \mathfrak{g}_{r_i}$, $f_i \in \mathfrak{g}_{-r_i}$.
$\alpha \in Q$ is a **root** of $\mathfrak{g}$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The root α is **positive** if α is a sum of simple roots. For any root α either α or $-\alpha$ is positive.
Let $\Phi = \Phi_+ \cup \Phi_-$ be the set of roots of $\mathfrak{g}$.
$\alpha \in \Phi$ is **real** if $\langle \alpha, \alpha \rangle > 0$, and **imaginary** if $\langle \alpha, \alpha \rangle \leq 0$.

Weyl group of Borcherds algebra

The Weyl group W of UBA $\mathfrak{g}$ is the group of isometries of the root lattice Q generated by reflections w_i corresponding to the simple real roots.

$$w_i(r_j) = r_j - 2\frac{\langle r_i, r_j \rangle}{\langle r_i, r_i \rangle} r_i = r_j - 2\frac{a_{ij}}{a_{ii}} r_i.$$

Recall that $2\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}$ since $a_{ii} > 0$. Let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{g}$ generated by the elements h_{ij} for all $i, j \in J$.

There is a homomorphism of abelian groups $Q \to \mathfrak{h}$: $r_i \mapsto h_i$ preserving scalar product. This homomorphism need not to be injective.

The **root system** and **Weyl group** of **any Borcherds algebra** $\mathfrak{g}$ is defined to be that of the corresponding universal Borcherds algebra.

Remarks on weight modules

Recall that if $\mathfrak{g}$ is a finite dimensional simple Lie algebra over $\mathbb{C}$, the irreducible finite dimensional $\mathfrak{g}$-modules are bijective to dominant integral weights. A weight $\lambda \in \mathfrak{h}^*$ is dominant and integral iff $\lambda(h_i) \geq 0$ and $\lambda(h_i) \in \mathbb{Z}$ $\forall i \in J$. λ arises as highest weight of this module where $\lambda, \mu \in \mathfrak{h}^*$ satisfy $\lambda \succ \mu$ iff $\lambda - \mu$ is a sum of simple roots.

Finite dimensional irreducible $\mathfrak{g}$-modules are also bijective to antidominant integral weights, i.e. which satisfy $\lambda(h_i) \leq 0$ and $\lambda(h_i) \in \mathbb{Z}$ $\forall i \in J$. There is a unique lowest weight for the module, and it is antidominant and integral. In the case of Borcherds algebras, it is most convenient to consider lowest weight modules rather than highest ones.

Weyl's and Kac's character formulas

Recall that if $\mathfrak{g}$ is a finite simple Lie algebra and λ is an antidominant integral weight, the corresponding finite dimensional irreducible lowest weight module M_λ has character given by the Weyl's character formula

$$\chi(M_\lambda) e^\rho \prod_{\alpha \in \Phi^+} (1 - e^\alpha) = \sum_{w \in W} \epsilon(w) w(e^{\lambda + \rho}),$$

where $\rho = -\sum_i \omega_i$, ω_i are positive real roots.

Next, suppose that $\mathfrak{g}$ is a SKMA and $\lambda \in \mathfrak{h}^*$ is a weight satisfying $\lambda(h_i) \leq 0$, $\lambda(h_i) \in \mathbb{Z}$ $\forall i \in J$. Then $\mathfrak{g}$ has a corresponding irreducible lowest module M_λ whose character os given by Kac's character formula

$$\chi(M_\lambda) e^\rho \prod_{\alpha \in \Phi} (1 - e^\alpha)^{mult \alpha} = \sum_{w \in W} \epsilon(w) w(e^{\lambda + \rho}),$$

where $\rho \in \mathfrak{h}^*$ is any element satisfying $\rho(h_i) = -1$ $\forall i \in J$. This time the sum and the product may be infinite.

Borcherds' character formula

Let $\mathfrak{g}$ be Borcherds algebra and let $\lambda \in Q \otimes \mathbb{R}$ satisfy
• $\langle \lambda, r_i \rangle \leq 0$ $\forall i \in J$; • $2\frac{\langle \lambda, r_i \rangle}{\langle r_i, r_i \rangle} \in \mathbb{Z}$ for all i s.t. $\langle r_i, r_i \rangle > 0$.
Then there is a corresponding irreducible lowest weight module M_λ with a character given by

$$\chi(M_\lambda) e^\rho \prod_{\alpha \in \Phi^+} (1 - e^\alpha)^{mult \alpha} = \sum_{w \in W} \epsilon(w) w \left(e^{\lambda + \rho} \sum_{\alpha \in Q} \epsilon(\alpha) e^\alpha \right),$$

where $\rho \in Q \otimes \mathbb{R}$ satisfies $\langle \rho, r_i \rangle = -\frac{1}{2} \langle r_i, r_i \rangle > 0$ $\forall i \in J$ s.t. $\langle r_i, r_i \rangle > 0$, and $\epsilon(\alpha) = (-1)^k$ if $\alpha \in Q$ is a sum of k orthogonal simple imaginary roots all orthogonal to λ, and $\epsilon(\alpha) = 0$ otherwise.

This formula reduces to Kac's character formula in the case of SKMA since in this case there are no simple imaginary roots and so

$$\sum_{\alpha \in Q} \epsilon(\alpha) e^\alpha = 1.$$

In the special case $\lambda = 0$, the module M_λ the module M_λ is the trivial 1-dimensional module and Borcherds' character formula becomes a **Borcherds denominator identity**:

$$e^\rho \prod_{\alpha \in \Phi^+} (1 - e^\alpha)^{mult \alpha} = \sum_{w \in W} \epsilon(w) w \left(e^\rho \sum_{\alpha \in Q} \epsilon(\alpha) e^\alpha \right),$$

where $\rho \in Q \otimes \mathbb{R}$ is any vector satisfying $\langle \rho, r_i \rangle = -\frac{1}{2} \langle r_i, r_i \rangle$ $\forall i \in J$.

Monster Lie Algebra

Start with the Monster Vertex Algebra $V^\natural$ which has a conformal vector of central charge 24. Replace it with a VOA of central charge 26.

Let Π be a lattice of rank 2 with a scalar product $\Pi \times \Pi \to \mathbb{Z}$ defined by $\langle b_1, b_1 \rangle = 0$, $\langle b_1, b_2 \rangle = -1$, $\langle b_2, b_2 \rangle = 0$, where b_1, b_2 is a basis of Π.

There is a vertex algebra V_Π associated with Π, with a conformal vector of central charge 2.

Form a tensor product $V^\natural \otimes V_\Pi$. This is a vertex operator algebra with conformal vector $w \otimes 1 + 1 \otimes w_\Pi$ of central charge 26, for w and w_Π being conformal vectors for $v^\natural$ and V_Π respectively.

Symmetric bilinear forms on $V^\natural$ and V_Π define symmetric bilinear form on $V^\natural \otimes V_\Pi$.

Define the subspaces

$$\begin{aligned} P^1 &= \{v \in V^\natural \otimes V_\Pi | L_0(v) = v, L_i(v) = 0, i \geq 1\}, \\ P^0 &= \{v \in V^\natural \otimes V_\Pi | L_0(v) = 0, L_i(v) = 0, i \geq 1\} \end{aligned}$$

The quotient $(V^\natural \otimes V_\Pi)/T(V^\natural \otimes V_\Pi)$ is a Lie algebra. [Borcherds R.E. Monstrous Moonshine and monstrous Lie superalgebras. Invent. Math. 109: 405 – 444, 1992]

The space $P^1/T(V^\natural \otimes V_\Pi) \cap P^1$ can be identified with a Lie subalgebra of $(V^\natural \otimes V_\Pi)/T(V^\natural \otimes V_\Pi)$. In fact, $TP^0 \subseteq P^1$ and $TP^0 = T(V^\natural \otimes V_\Pi) \cap P^1$. Thus, P^1/DP^0 has the structure of a Lie algebra.

The symmetric bilinear form on $V^\natural \otimes V_\Pi$ induces one on P^1, and TP^0 lies in the radical of the induced form. Thus we obtain a symmetric bilinear form on the Lie algebra P^1/TP^0.
Define

$$\mathfrak{M} = \frac{P^1/TP^0}{\mathrm{rad}\langle\cdot,\cdot\rangle}$$

This is a Lie algebra called the **Monster Lie algebra**.

The vertex algebra V_Π has a grading by elements of the lattice Π and induces gradings on $V^\natural \otimes V_\Pi$ and on subquotient $\mathfrak{M}$ by elements of Π:

$$\mathfrak{M} = \bigoplus_{m,n \in \mathbb{Z}} \mathfrak{M}_{(n,m)},$$

where (m, n) is the graded component corresponding to $mb_1 + nb_2 \in \Pi$.

No-Ghost Theorem

Theorem. Suppose that

✓ V be a vector space with a nonsingular bilinear form $\langle\cdot,\cdot\rangle$, $\mathfrak{v}$ the Virasoro algebra, $\mathfrak{v} \curvearrowright V$ in such a way that
- the adjoint of L_i is L_{-i};
- the central element of the Virasoro algebra acts as multiplication by 24;
- any vector of V is a sum of eigenvectors of L_0 with nonnegative integral eigenvalues;
- all the eigenspaces of L_0 are finite dimensional.

✓ V_{i-1} be the subspace of V on which L_0 has eigenvalue i.
✓ G is a group, $G \curvearrowright V$ preserves all this structure.
✓ V_Π be the vertex algebra of the 2-dimensional even lattice Π (so that V_Π is Π-graded, has a bilinear form $\langle\cdot,\cdot\rangle$, and $\mathfrak{v} \curvearrowright V_\Pi$).
✓ P^1 be the subspace as defined before, and we let P^1_α be the subspace of P^1 of degree $\alpha \in \Pi$.
✓ All these spaces inherit an action of G from the action $G \curvearrowright V$ and the trivial action of G on V_Π and $\mathbb{R}^2$.
Then the quotient of P^1_α by the nullspace of its bilinear form is naturally isomorphic, as a G-module with an invariant bilinear form, to

$$\begin{cases} V_{-\langle\alpha,\alpha\rangle/2} & \text{if } \alpha \neq 0, \\ V_0 \oplus \mathbb{R}^2 & \text{if } \alpha = 0 \end{cases}$$

Remark. In the original statement of this theorem [Goddard P., Thorn C. B. Compatibility of the dual Pomeron with unitarity and the absence of ghosts in the dual resonance model. Phys. Lett., B 40, No.2: 235 – 238 (1972)] V was part of underlying vector space of vertex algebra of a positive definite lattice, so the inner product on V_{i-1} was positive definite, and thus, P^1_α had no vectors of negative norm ("ghosts") for $\alpha \neq 0$.

Sketch of a Proof [Borcherds R.E. Monstrous moonshine and monstrous Lie superalgebras. Invent. Math., 109: 405 – 444, 1992], [Goddard P., Thorn C. B. Compatibility of the dual Pomeron with unitarity and the absence of ghosts in the dual resonance model. Phys. Lett., B 40, No.2: 235 – 238 (1972)]
Fix nonzero $\alpha \in \Pi$ and some norm 0 vector $w \in \Pi$ with $\langle\alpha, w\rangle \neq 0$. There is an action $\mathfrak{v} \curvearrowright V \otimes V_\Pi$ with operators $L_i \in \mathfrak{v}$ satisfying

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{1}{2}\begin{pmatrix} i+1 \\ 3 \end{pmatrix}\delta_{i+j,0}26,$$

and the adjoint of L_i is L_{-i}.
Define operators K_i, for $i \in \mathbb{Z}$, by $K_i = v_{i-1}$ where $v = e^{-w}_{-2}e^w$ in the vertex algebra of Π, and $e^w \in \mathbb{R}[\Pi]$ corresponds to $w \in \Pi$, and e^{-w} its inverse. K_i satisfy $[L_i, K_j] = -jK_{i+j}$. $[K_i, K_j] = 0$, since w has norm 0 and the adjoint of K_i is K_{-i}.

Define following subspaces in $V \otimes V_\Pi$:
- H – subspace of degree $\alpha \in \Pi$, $H^1 = \{h \in H | L_0(h) = h\}$.
- $P = \{h \in H | L_i(h) = 0 \forall i > 0\}$, $P^1 := H^1 \cap P$.
- S – "space of spurious vectors", $S = \{h \in H | h \perp P\}$, $S^1 := H^1 \cap S$.
- $N = S \cap P$ is the radical of the bilinear form on P, and $N^1 = H^1 \cap N$.
- T – "transverse space", $T = \{p \in P | K_i p = 0, i > 0\}$, $T^1 = H^1 \cap T$.
- K is the space generated by the action of the operators $K_i, i > 0$.

• $Ve^\alpha = V \otimes e^\alpha \subset H$.
There are inclusions of subspaces in H:

$$\begin{tikzcd}
S & & P & & K & \\
& N \arrow[ul, hook] \arrow[ur, hook] & & T \arrow[ul, hook] \arrow[ur, hook] & & Ve^\alpha \arrow[ul, hook]
\end{tikzcd}$$

The isomorphism $V_{-\langle\alpha,\alpha\rangle/2} \cong P^1/(N \cap P^1)$ is done by zigzagging up and down. We show that 1) Ve^α and T are both isomorphic to K mod its nullspace; 2) T^1 is isomorphic to P^1 mod its nullspace $P^1 \cap N$.

The theorem follows from the sequence of lemmas

Lemma 1. If f is a vector of nonzero norm in T, then the vectors of the form $L_{m_1} L_{m_2} \dots K_{n_1} K_{n_2} \dots (f)$ for all sequences of integers $0 > m_1 \geq m_2 \geq \dots$, $0 > n_1 \geq n_2 \geq \dots$, are linearly independent and span a space invariant under the operators K_i and L_i, on which the bilinear form is nonsingular.

Lemma 2. The bilinear form on T is nonsingular, and K is the direct sum of T and the nullspace of K.

Lemma 3. Ve^α is naturally isomorphic to T.

Lemma 4. The associative algebra generated by the elements L_i for $i < 0$, is generated by elements which map S^1 into S.

Lemma 5. $P^1 = T^1 \oplus N^1$.

Applying the no-ghost theorem to the vertex algebra

$$\mathfrak{M} = \bigoplus_{m,n \in \mathbb{Z}} \mathfrak{M}_{m,n}$$

(where m, n mark the graded component corresponding to $mb_1 + nb_2 \in \Pi$) we conclude that for $\alpha \in \Pi$ $\mathfrak{M}_\alpha \cong V^\natural_{-\langle\alpha,\alpha\rangle/2}$ for $\alpha \neq 0$, where

$$V^\natural_i = \{v \in V^\natural | L_0(v) = (i+1)v\}.$$

Let $\alpha = mb_1 + nb_2 \in \Pi$. Since $\langle\alpha,\alpha\rangle = -2mn$, then $\mathfrak{M}_{m.n} \cong V^\natural_{m,n}$ if $\alpha \neq (0,0)$. The no-ghost theorem asserts that $\mathfrak{M}_{0,0} \cong \mathbb{R}^2$.

The graded components of the monster Lie algebra $\mathfrak{M}$ are as follows

$$\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & V^\natural_4 & V^\natural_8 & V^\natural_{12} & V^\natural_{16} & \\
 & 0 & 0 & 0 & 0 & 0 & V^\natural_3 & V^\natural_6 & V^\natural_9 & V^\natural_{12} & \\
 & 0 & 0 & 0 & 0 & 0 & V^\natural_2 & V^\natural_4 & V^\natural_6 & V^\natural_8 & \\
 & 0 & 0 & 0 & V^\natural_{-1} & 0 & V^\natural_1 & V^\natural_2 & V^\natural_3 & V^\natural_4 & \\
\cdots & 0 & 0 & 0 & 0 & \mathbb{R}^2 & 0 & 0 & 0 & 0 & \cdots \\
 & V^\natural_4 & V^\natural_3 & V^\natural_2 & V^\natural_1 & 0 & V^\natural_{-1} & 0 & 0 & 0 & \\
 & V^\natural_8 & V^\natural_6 & V^\natural_4 & V^\natural_2 & 0 & 0 & 0 & 0 & 0 & \\
 & V^\natural_{12} & V^\natural_9 & V^\natural_6 & V^\natural_3 & 0 & 0 & 0 & 0 & 0 & \\
 & V^\natural_{16} & V^\natural_{12} & V^\natural_8 & V^\natural_4 & 0 & 0 & 0 & 0 & 0 & \\
 & & & & & \vdots & & & & &
\end{array}$$

The group ring $\mathbb{R}[\Pi]$ of the lattice Π has an involution

$$e^\alpha \mapsto (-1)^{\langle\alpha,\alpha\rangle/2} e(-\alpha) \text{for } \alpha \in \Pi.$$

It induces an involution on the vertex algebra $V_\Pi = S(\widetilde{\mathfrak{h}}^-) \otimes \mathbb{R}[\Pi]$ and hence an involution on $V^\natural \otimes V_\Pi$, which acts trivially on $V^\natural$. Acting on a subquotient $\mathfrak{M}$ of $V^\natural \otimes V_\Pi$ it gives a map $\omega : \mathfrak{M} \to \mathfrak{M}$ s.t.
• $\omega^2 = 1$; • $\omega \mathfrak{M}_{m,n} = \mathfrak{M}_{-m,-n}$; • $\omega = -1$ on $\mathfrak{M}_{0,0}$; • $\langle\omega x, \omega y\rangle = \langle x, y\rangle$;
where $\langle\cdot,\cdot\rangle$ is the invariant bilinear form on $\mathfrak{M}$. Moreover, the contravariant form $\langle x, y\rangle_0 = -\langle x, \omega y\rangle$ for $x, y \in \mathfrak{M}$ is positive definite on $\mathfrak{M}_{m,n}$ for all $(m,n) \neq (0,0)$.
Give a $\mathbb{Z}$-grading to $\mathfrak{M}$ by the formula $\deg \mathfrak{M}_{m,n} = 2m + n$. Then $\mathbb{Z}$-graded components are

$$\begin{array}{ccccccccccccc}
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & V^\natural_2 \oplus V^\natural_3 & V^\natural_2 & V^\natural_1 & 0 & V^\natural_{-1} & \mathbb{R}^2 & V^\natural_{-1} & 0 & V^\natural_1 & V^\natural_2 & V^\natural_2 \oplus V^\natural_3 & \cdots
\end{array}$$

Hence $\mathfrak{M}$ satisfies the axioms for a Borcherds algebra.

Let Q be a root lattice of the Borcherds algebra $\mathfrak{M}$, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{M}$.
Then $\mathfrak{h} = \mathfrak{M}_{0,0} = \mathbb{R}^2$, and we have a map

$$Q \to \mathfrak{h}.$$

This map takes any simple root $r_i \in Q$ to $h_i \in \mathfrak{h}$, and preserves the scalar product. Elements of $\mathfrak{h}$ can be written in the form $mb_1 + nb_2$, $m, n \in \mathbb{Z}$. The images of simple roots in Q equal to

$$(1,-1),(1,1),(1,2),(1,3),(1,4),\ldots.$$

Thus $\mathfrak{M}$ has infinitely many simple roots (they are not linearly independent). Since $\langle mb_1+nb_2, mb_1+nb_2\rangle = -2mn$, $(1,-1)$ gives real simple root and other simple roots $(1,n)$, $n \geq 1$, are imaginary.
There can be several simple roots mapping to a given $(b_1 + nb_2) \in \mathfrak{h}$. The number of such elements in a **multiplicity** of $(1,n)$.

The multiplicity of $(1,n)$ equals

$$\dim \mathfrak{M}_{1,n} = \dim V_n^\natural = c_n,$$

where c_n is the coefficient of the normalized Hauptmodul

$$J(z) = q^{-1} + \sum_{n\geq 1} c_n q^n, \quad q = e^{2\pi i z}.$$

Thus
- (1,-1) has multiplicity 1,
- (1,1) has multiplicity 196 884,
- (1,2) has multiplicity 21 493 760,
- ...

and the sum of the simple root spaces in $\mathfrak{M}$ is isomorphic to the Moonshine module $V^\natural$.
Hence, the Monster Lie algebra $\mathfrak{M}$ is contained in the Monster vertex algebra $V^\natural$ as the sum of its root spaces corresponding to simple roots.

The symmetric matrix (a_{ij}) corresponding to the Borcherds algebra $\mathfrak{M}$ is thus a countable matrix with many repeated rows and columns.
Since $\mathfrak{M}$ has only one simple root, its Weyl group has order 2. Any root of $\mathfrak{M}$ maps to an element $mb_1 + nb_2 \in \mathfrak{h}$ s.t. $\mathfrak{M}_{0,0} \neq 0$ and $(m,n) \neq (0,0)$. The multiplicity of (m,n) is then $\dim \mathfrak{M}_{m,n} = \dim V_{mn}^\natural = c_{mn}$.

Denominator identities for Monster Lie algebra

Consider the homomorphism $Q \to \mathfrak{h}$ sending simple roots to $\mathfrak{h}$. We call images of roots in $\mathfrak{h}$ as roots also. For Monster Lie algebra $\mathfrak{M}$ simple roots in $\mathfrak{h}$ are $mb_1 + nb_2 \in \mathbb{Z}$, $m, n \in \mathbb{Z}$ and $mn > 0$ or $mn = -1$. If there are k roots in Q mapping to the same root in $\mathfrak{h}$, this root in $\mathfrak{h}$ has multiplicity k. We know that simple roots of $\mathfrak{M}$ are $(1,-1),(1,1),(1,2),(1,3),\ldots$, and we could take $\rho = (-1,0)$ because $\langle(-1,1),(1,n)\rangle = n$, $\langle(1,n),(1,n)\rangle = -2n$ for all n, so that $\langle \rho, r_i\rangle = -\frac{1}{2}\langle r_i, r_i\rangle$ for each simple root r_i. Also we know that the root (m,n) has multiplicity exactly c_{mn}.
Let $p = e^{(1,0)}$ and $q = e^{(0,1)}$. We have $e^\rho = e^{-(1,0)} = p^{-1}$, so left hand side of the Borcherds identity becomes

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c_{mn}}.$$

remember also that for $\alpha \in Q$ $\epsilon(\alpha) = (-1)^k$ if α is the sum of k imaginary orthogonal simple roots, and $\epsilon(\alpha) = 0$ otherwise.

For the Monster Lie algebra $\mathfrak{M}$ there are no two imaginary simple roots, because $\langle(1,m),(1,n)\rangle = -m-n < 0\ \forall m,n \in \mathbb{Z}$. Thus the elements $\alpha \in Q$ contributing to the sum $\sum_\alpha \epsilon(\alpha)e^\alpha$ are $\alpha = 0$ with $\epsilon(\alpha) = 1$, and all imaginary simple roots $(1,n) \in \mathfrak{h}$, $n \in \mathbb{N}$. Since there are precisely c_n of these roots in Q (mapping to $(1,n)$) and all of them have $\epsilon(\alpha) = -1$,

$$\sum_{\alpha\in Q} \epsilon(\alpha)e^\alpha = 1 - \sum_{n>0} c_n p q^n.$$

Also $|W| = 2$ and $W = \{1, s\}$, $s(p) = q$, $s(q) = p$. Thus right hand side of the Borcherds identity is

$$\sum_{w\in W} e(w)w\left(e^\rho \sum_{\alpha\in Q}\epsilon(\alpha)e^\alpha\right) = \sum_{w\in W} e(w)w\left(p^{-1}\left(1 - \sum_{n>0} c_n p q^n\right)\right) =$$

$$\left(p^{-1} - \sum_{n>0} c_n p^n\right) - \left(q^{-1} - \sum_{n>0} c_n q^n\right) = j(p) - j(q).$$

Combining left hand side and right hand side one gets **Zagier's identity**

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c_{mn}} = j(p) - j(q).$$

Twisted denominator identity

To complete the proof of the Moonshine conjecture we need a generalization if the Zagier's identity.

Let U be a finite-dimensional $\mathbb{R}$-vector space with a graded decomposition

$$U = \bigoplus_{\alpha \in L} U_\alpha$$

for a lattice L.

The **graded dimension** if U is $gr\dim U = \sum_{\alpha\in L}(\dim U_\alpha)e^\alpha \in \mathbb{R}[L]$ where $\mathbb{R}[L]$ is a group algebra of L with basis L, for $\alpha \in L$. Let $\bigwedge^k U^\vee = \{k\text{-linear alternate forms } \omega : U \times \cdots \times U \to \mathbb{R}\}$, for $k > 0$ and $\bigwedge^0 U^\vee = \mathbb{R}$. One has

$$\sum_{k\geq 0}(-1)^k gr\dim \bigwedge^k U^\vee = \prod_{\alpha\in L}(1-e^\alpha)^{\dim U_\alpha}. \tag{3}$$

The right hand side can be written as $\exp\left(-\sum_{k>0}\frac{1}{k}\sum_{\alpha\in L}(\dim U_\alpha)e^{k\alpha}\right)$ (this can be verified moving backwards and using the Taylor expansion of $\log(1+x)$).

Let G be a finite group and U be a G-module s.t. $G \curvearrowright U_\alpha$ $\forall \alpha \in L$.

The **graded character** of a G-module U is

$$gr\chi U : G \to \mathbb{R}[L] : g \mapsto \sum_\alpha trace(g|U_\alpha)e^\alpha.$$

Then the alternate sum $\sum_{k\geq 0}(-1)^k gr\chi \bigwedge^k U^\vee$ is given by the map

$$g \mapsto \exp\left(-\sum_{k>0}\frac{1}{k}\sum_{\alpha\in L} trace(g^k|U_\alpha)e^{k\alpha}\right).$$

When $g = 1$ this map reduces to the dimension formula (3). If U is an infinite dimensional vector space and $U = \bigoplus_{\alpha\in L} U_\alpha$ is its graded decomposition s.t. each graded component is finite dimensional, then the formulae are still valid.

Let $\mathfrak{g}$ be a Borcherds algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ where $\mathfrak{n}^\pm = \sum_{\alpha\in\Phi^\pm}\mathfrak{g}_\alpha$ ($\mathfrak{h}^+$ can be infinite dimensional but each $\mathfrak{g}_\alpha$ is finite dimensional). Consider a complex of vector spaces

$$\cdots \xrightarrow{d_4} \bigwedge^3(\mathfrak{n}^+)^\vee \xrightarrow{d_3} \bigwedge^2(\mathfrak{n}^+)^\vee \xrightarrow{d_2} \bigwedge^1(\mathfrak{n}^+)^\vee \xrightarrow{d_1} \bigwedge^0(\mathfrak{n}^+)^\vee \xrightarrow{d_0} 0$$

with homology groups

$$H_k\mathfrak{n}^+ = \frac{\ker d_k}{\operatorname{im} d_{k+1}}, k \geq 0.$$

Here $\bigwedge^k(\mathfrak{n}^+)^\vee$ and $H_k\mathfrak{n}^+$ are graded vector spaces with finite dimensional graded components. Obviously,

$$\sum_{k\geq 0}(-1)^k gr\dim \bigwedge^k(\mathfrak{n}^+)^\vee = \sum_{k\geq 0}(-1)^k gr\dim H_k\mathfrak{n}^+.$$

Garland and Lepowky proved that for KMA (and also it is true for Borcherds algebras) $H_k\mathfrak{n}^+$ can be identified with a subspace of $\bigwedge^k(\mathfrak{n}^+)^\vee$ as follows:

$$H_k\mathfrak{n}^+ = \begin{cases} ((\bigwedge^k(\mathfrak{n}^+)^\vee)_\alpha & \text{if } \langle\alpha+\rho,\alpha+\rho\rangle = \langle\rho,\rho\rangle, \\ 0 & \text{in other case} \end{cases}$$

This holds for each α in the root lattice of $\mathfrak{g}$.

For the monster Lie algebra $\mathfrak{g} = \mathfrak{M}$ the graded dimension formula takes the view

$$\sum_{k>0}(-1)^k gr\dim \bigwedge^k \mathfrak{M}^+ = \prod_{(m,n),m>0}(1-p^m q^n)^{c_{mn}}$$

where $p = e^{(1,0)}$, $q = e^{(0,1)}$ and $\mathfrak{M}^+ = \sum_{\alpha\in\Phi^+}\mathfrak{M}_\alpha$.

We compute $\sum_{k\geq 0}(-1)^k gr\dim H_k\mathfrak{M}^+$:

$gr\dim H_0\mathfrak{M}^+ = e^0$ i.e. $\bigwedge^0\mathfrak{M}^+ = \mathbb{R}$ is 1-dimensional with weight e^0 since $\langle 0+\rho, 0+\rho\rangle = \langle\rho,\rho\rangle$.

Next, direct computations lead to

$$gr\dim H_1\mathfrak{M}^+ = \sum_{n\in\mathbb{Z}} c_n p q^n,$$

$$grdim\, H_2\mathfrak{M}^+ = \sum_{m\geq 2} c_{m-1}p^m,$$

$$grdim\, H_k\mathfrak{M}^+ = 0$$

for $k \geq 3$. Thus we have

$$\sum_{k\geq 0}(-1)^k grdim\, H_k\mathfrak{M}^+ = e^0 - \sum_{n\in\mathbb{Z}} c_n pq^n + \sum_{m\geq 2} c_{m-1}p^m =$$

$$e^0 + p\sum_{m\geq 1} c_m p^m - p\sum_{n\in\mathbb{Z}} c_n q^n = e^0 + p\left(\sum_{m\in\mathbb{Z}} c_m p^m - p^{-1}\right) - p\sum_{n\in\mathbb{Z}} c_n q^n =$$

$$p(j(p) - j(q))$$

and we have derived the denominator identity.

In order to obtain the twisted denominator identity, consider $\mathfrak{M}^+$ as an $\mathbb{M}$-module for the Monster group $\mathbb{M}$. Start from the equality

$$\sum_{k\geq 0}(-1)^k grdim \bigwedge^k (\mathfrak{M}^+)^\vee = \sum_{k\geq 0}(-1)^k grdim\, H_k(\mathfrak{M}^+).$$

The left hand side is the map

$$g \mapsto exp\left(-\sum_{k>0}\frac{1}{k}\sum_{\alpha\in\Phi^+} trace(g^k|(\mathfrak{M}^+)_\alpha)e^{k\alpha}\right)$$

for $g \in \mathbb{M}$. Replacing all dimensions by characters in the computations of $H_k\mathfrak{M}^+$, we get

$$\sum_k gr\chi H_k\mathfrak{m}^+ : g \mapsto p\left(\sum_{n\in\mathbb{Z}} trace(g|V_n^\natural)p^n - \sum_{n\in\mathbb{Z}} trace(g|V_n^\natural)q^n\right).$$

Comparing two recent formulae we get **twisted denominator identity for the monster Lie algebra** $\mathfrak{M}$:

$$p^{-1}exp\left(-\sum_{k>0}\frac{1}{k}\sum_{(m,n),m>0} trace(g^k|V_{mn}^\natural p^{mk}q^{nk})\right) =$$
$$\sum_{n\in\mathbb{Z}} trace(g|V_n^\natural)p^n - \sum_{n\in\mathbb{Z}} trace(g|V_n^\natural)q^n.$$

Denoting $c_g(n) := trace(g|V_n^\natural)$ we get

$$p^{-1}exp\left(-\sum_{k>0}\frac{1}{k}\sum_{(m,n),m>0} c_{g^k}(mn)p^{mk}q^{nk})\right) = \sum_{n\in\mathbb{Z}} c_g(n)p^n - \sum_{n\in\mathbb{Z}} c_g(n)q^n.$$

Replication formulae
By comparing the coefficients of p^2 and p^4 in the twisted denominator identity Borcherds derived the replication formulae.

To complete the proof, it remains to show that the coefficients $c_g(1)$, $c_g(2)$, $c_g(3)$ and $c_g(5)$ of the McKay – Thompson series $T_g(z)$ agree with right hand sides of (2). To obtain the coefficients for the graded characters $T_g(z)$ it is sufficient to know how the modules V_1, V_2, V_3 and V_5 for the Monster $\mathbb{M}$, with dimensions c_1, c_2, c_3 and c_5 respectively, decompose into irreducible modules. The only irreducible characters of $\mathbb{M}$ less then or equal to $c_5 = \dim V_5$ are $\chi_0, \chi_1, \ldots, \chi_6$, thus these are the only possible irreducible components of V_1, V_2, V_3, V_5. Borcherds proved that $\dim V_1 = \chi_0 + \chi_1$, $\dim V_2 = \chi_0 + \chi_1 + \chi_2$, $\dim V_3 = 2\chi_0 + 2\chi_1 + \chi_2 + \chi_3$, $\dim V_5 = 4\chi_0 + 5\chi_1 + 3\chi_2 + 2\chi_3 + \chi_4 + \chi_5 + \chi_6$, where $\chi_0, \ldots, \chi_6$ are first 6 irreducible characters of $\mathbb{M}$. This was proved by finding 7 elements $g_1, \ldots g_7$ for which the 7×7-matrix $\chi_i(g_j)$ is nonsingular and by showing that the above equations hold for each g_i. Then they hold for all $g \in \mathbb{M}$.

Remarks on a group
Not only the Monster but other sporadic simple groups were discussed including the Baby Monster $\mathbb{B}$, the Conway group Co_1, the Fisher group Fi'_{24}, the Harada – Norton group HN, the Held group He, and the Mathieu group M_{12}. Denominator identities for these groups are also obtained. Thus the Monstrous Moonshine is not restricted by the Monster group $\mathbb{M}$.

Why genus 0?
For a formal series $f(z) = q^{-1} + \sum_{n=1}^\infty b_n q^n$, an **order-$n$ modular equation for** f is a monic polynomial $F_n(x, y)$, $\deg F_n = n\prod_{primes\, p|n}(1 + 1/p)$ s.t.

$$F_n\left(f(z), f\left(\frac{az+b}{d}\right)\right) = 0$$

for all $a, b, d \in \mathbb{Z}$, $ad = n$, $\gcd(a, b, d) = 1$, and $0 \leq b < d$.

The degree $\deg F_n$ is precisely the the number of triples a, b, d. This triples come from the coset expansion

$$\Gamma_0(K)\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(K) = \bigcup_{a,b,d}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\Gamma_0(K)$$

for any K obeying $n \equiv 1(\mathrm{mod}\, K)$.

Theorem. Let $f(z)$ be a formal series $q^{-1} + \sum_{n=1}^{\infty} b_n q^n$, $b_i \in \mathbb{C}$. Suppose f satisfies a modular equation of order n for all $n \equiv 1(\mathrm{mod}\, N)$. Then
(a) f converges to a holomorphic function on H,
(b) if the symmetry group $\Gamma(f) := \{g \in SL_2(\mathbb{R}) | f(gz) = f(z)\}$ consists only of translations $\pm\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, then $f(z) = q^{-1} + \xi q$ for some $\xi \in \mathbb{C}$. If ξ is an algebraic number, then $\xi = 0$ or $\xi^{\gcd(24,N)} = 1$;
(c) if the symmetry group does not only contains translations, then $\Gamma(f)$ is genus 0 and f is a Hauptmodul for $\Gamma(f)$. Moreover, $\Gamma(f)$ contains some subgroup $\Gamma_0(K)$ for $K|N^\infty$.

$K|N^\infty$ means that all primes dividing K also divide N.

Conjecture.[Cohn, McKay, Cummins] Let $q^{-1}+\sum_{n=1}^{\infty} b_n q^n$ be a formal series and p, p' are distinct primes. If f satisfies modular equations for both p and p', then f converges in H to a holomorphic function, and either $f(z) = q^{-1} + \xi q$ for $\xi^{\gcd(p-1,p'-1)+1} = \xi$, or f is the Hauptmodul for a genus-0 group containing $\Gamma(N)$ for N coprime to pp'.

Strenghtening Moonshine

1987 Norton proposed a strengthening of the Monstrous Moonshine Conjecture. Among the assertions is the existence of the rule that produces special modular functions (Hauptmoduls) from commuting pairs of elements of the Monster.

The Borcherds – Höhn program proposed a way to obtain such a rule by constructing infinite-dimensional Lie algebras attached to elements of the Monster. These Lie algebras are expacted to manifest as algebras physical states in an orbifold conformal field theory (yet to be fully constructed with symmetries given by the Monster).

Umbral Moonshine

Cheng, Duncan, Harvey: M_{24} moonshine and others; in whole 23 moonshines relating groups to mock modular forms.

They conjectured that for each of those moonshines there is a string theory like the one for Monstrous moonshine, in which he mock modular form counts string states, and the group captures the model's symmetry.

A mock modular form always has an associated modular function named the "shadow": they named their conjectures as Umbral Moonshine.

Many of the mock modular forms that appear in the conjecture are among the 17 special examples in the paper of Ramanujan about "mock theta functions".

Umbral moonshine is related to 23 Niemeier lattices. [Cheng M., Duncan J.,& Harvey J. Research in Math. Sci., 1:3, 2014; Comm. Number Theory and Phys., 8, 2014].

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