Synchronizing Finite Automata

II-III. The Road Coloring Problem

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Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

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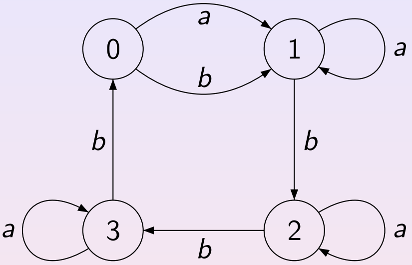

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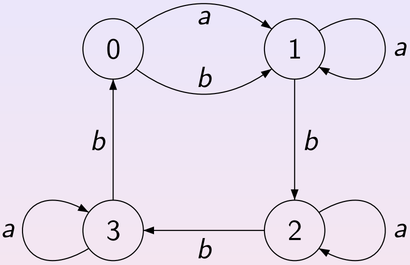

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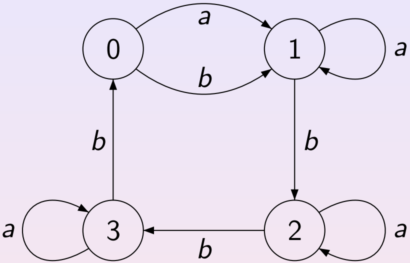

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Studying synchronizing automata, it is natural to restrict to the strongly connected case. For instance, it suffices to prove the Černý conjecture for this case, the general case would be an easy consequence.

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a synchronizing automaton with n states. Consider the set S of all states to which $\mathscr{A}$ can be synchronized and let $m = |S|$. If $q \in S$, then there exists a reset word $w \in \Sigma^*$ such that $Q.w = \{q\}$. For each $a \in \Sigma$, we have $Q.wa = \{q \,.\, a\}$ whence wa also is a reset word and $\delta(q, a) \in S$. Thus, restricting the function δ to $S \times \Sigma$, we get a subautomaton $\mathscr{S}$ with the state set S. Obviously, $\mathscr{S}$ is synchronizing and strongly connected.

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If the Černý conjecture holds true for strongly connected synchronizing automata, $\mathscr{S}$ has a reset word v of length $(m-1)^2$.

Now consider the partition π of Q into $n-m+1$ classes one of which is S and all others are singletons. Then π is a congruence of the automaton $\mathscr{A}$.

We recall the notion of a congruence and the related notion of the quotient automaton w.r.t. a congruence in the next slide. They will be essentially used in this lecture!

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Congruences and Quotient Automata

An equivalence π on the state set Q of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called a congruence if $(p, q) \in \pi$ implies $\big(\delta(p, a), \delta(q, a)\big) \in \pi$ for all $p, q \in Q$ and all $a \in \Sigma$. For π being a congruence, $[q]_\pi$ is the π-class containing the state q.

The *quotient* $\mathscr{A}/\pi$ is the DFA $\langle Q/\pi, \Sigma, \delta_\pi \rangle$ where $Q/\pi = \{[q]_\pi \mid q \in Q\}$ and the function δ_π is defined by the rule $\delta_\pi([q]_\pi, a) = [\delta(q, a)]_\pi$.

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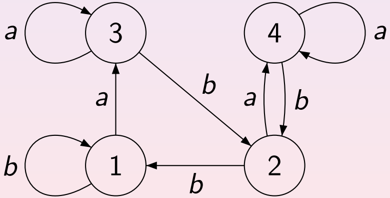

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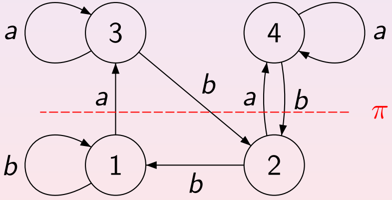

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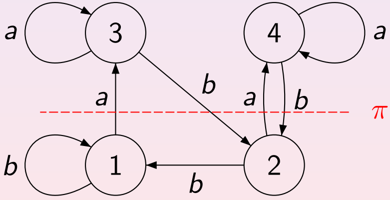

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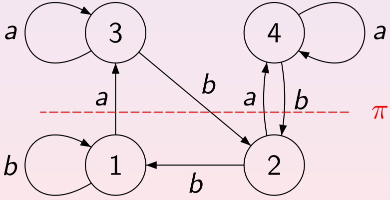

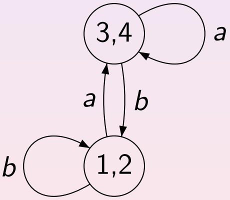

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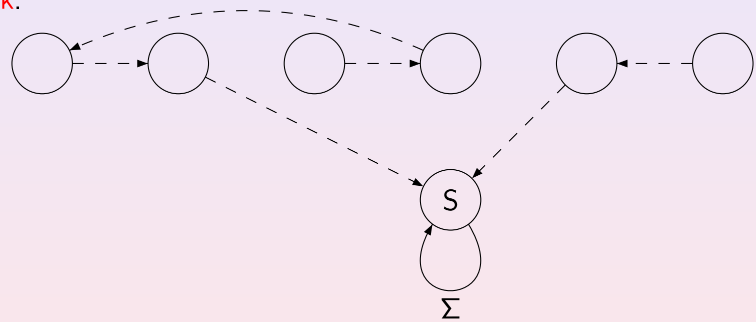

Automata with a Unique Sink

If a synchronizing automaton with k states has a unique sink, then it has a reset word of length $\leq \frac{k(k-1)}{2}$.

The algorithm makes at most $k-1$ steps and the length of the segment added in the step when t states still hold coins $(k-1 \geq t \geq 1)$ is at most $k-t$. The total length is $\leq 1+2+\cdots+(k-1) = \frac{k(k-1)}{2}$.

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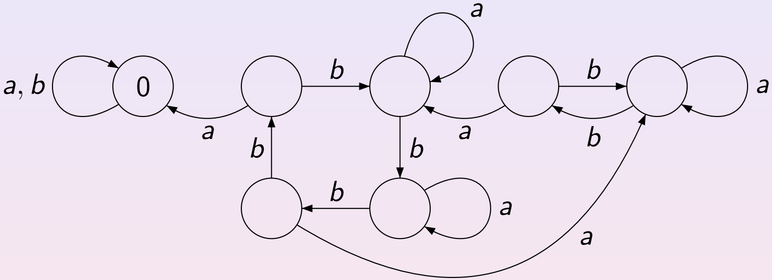

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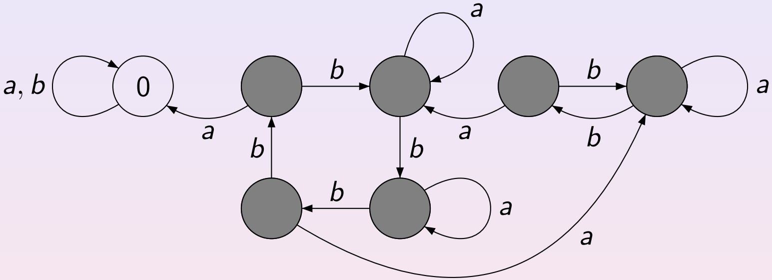

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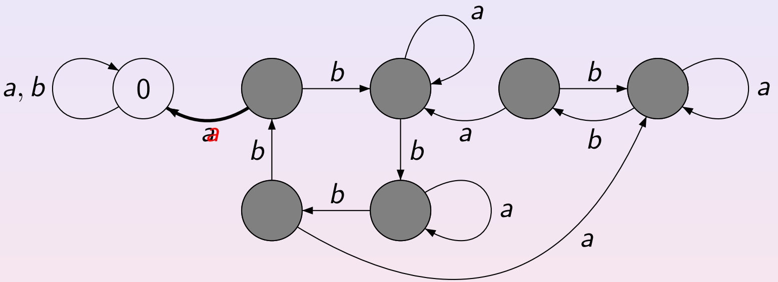

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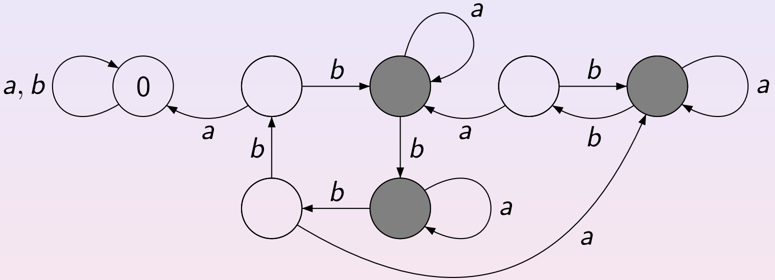

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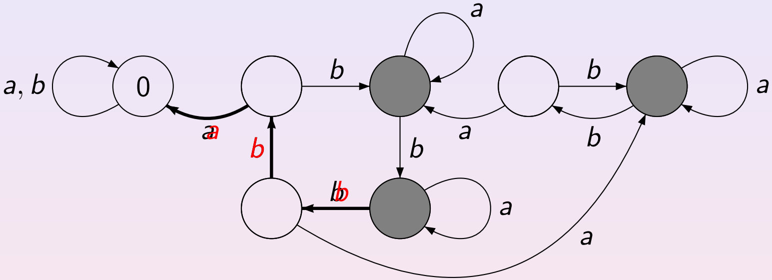

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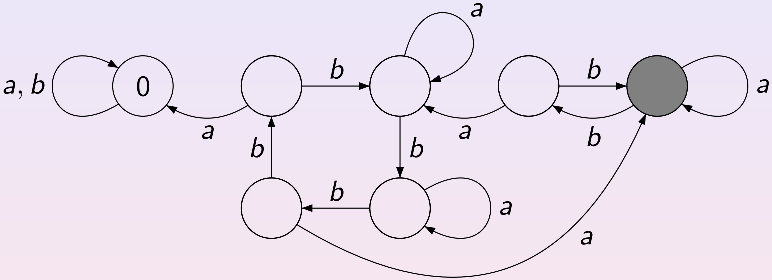

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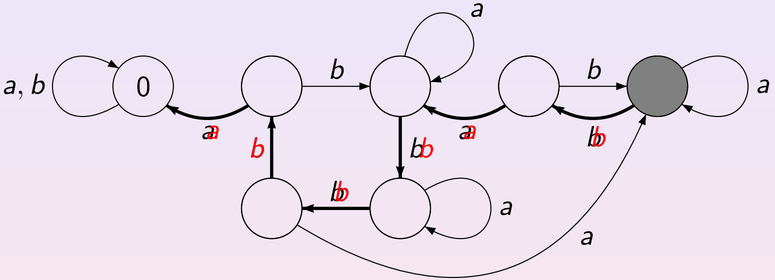

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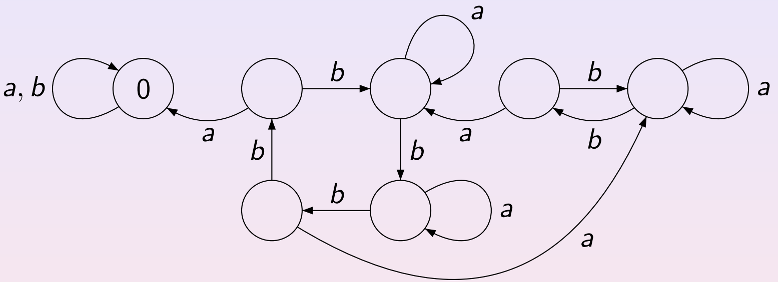

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Return to our reasoning: the quotient $\mathscr{A}/\pi$ is synchronizing with a unique sink and has $n-m+1$ states. Hence, $\mathscr{A}/\pi$ has a reset word u of length $\frac{(n-m+1)(n-m)}{2}$. Then $Q\,.\,u \subseteq S$.
Recall that we have assumed that the automaton $\mathscr{S}$ has a reset word v of length $(m-1)^2$. Then $S.v$ is a singleton, whence also $Q.uv \subseteq S.v$ is a singleton. Thus, uv is a reset word for $\mathscr{A}$, and the length of this word does not exceed

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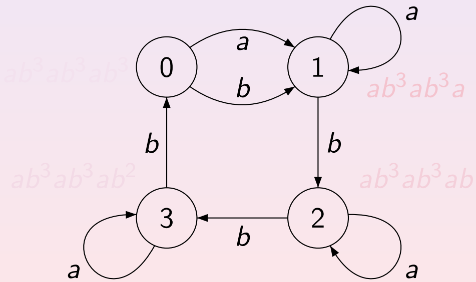

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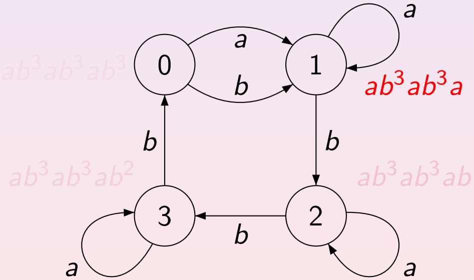

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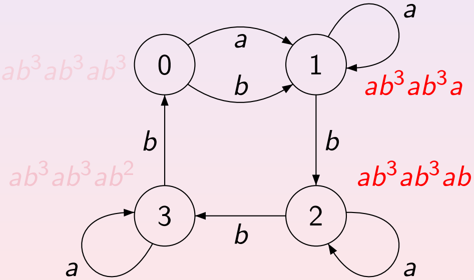

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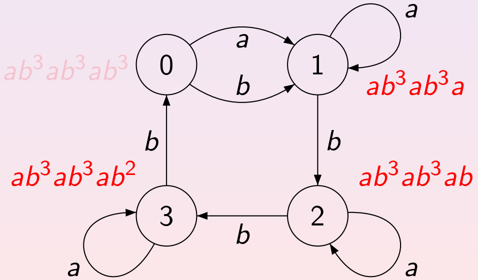

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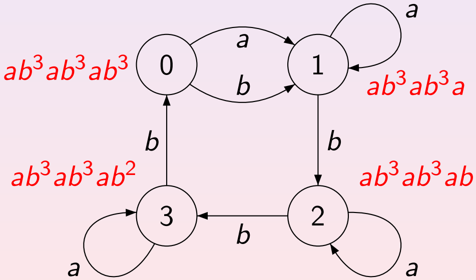
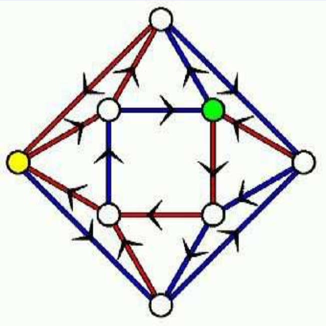


Example

Now think of the automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.



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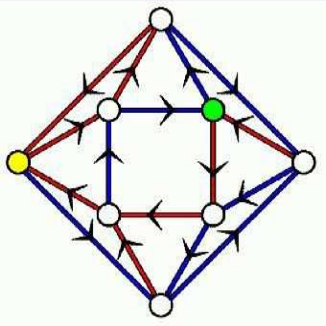

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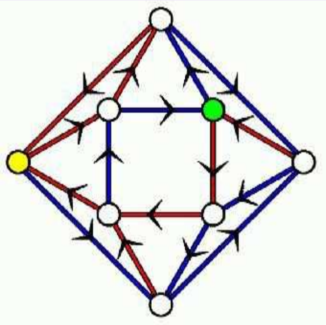

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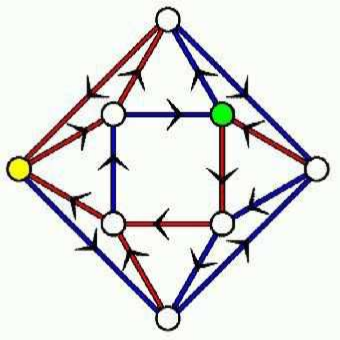

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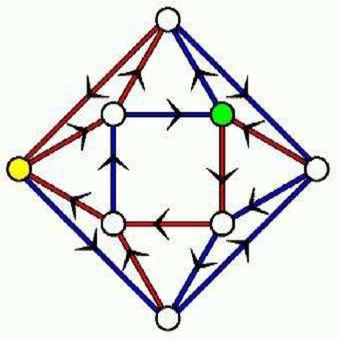

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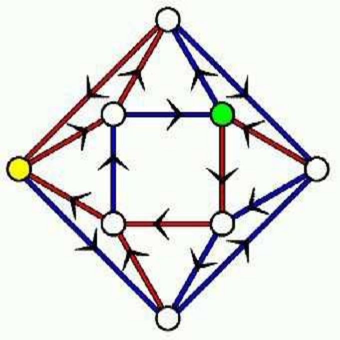

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We aim to help people to orientate in it, and as we have seen, a neat solution may consist in coloring the roads such that our digraph becomes a synchronizing automaton. When is such a coloring possible?

In other words: which strongly connected digraphs may appear as underlying digraphs of synchronizing automata?

An obvious necessary condition:

all vertices should have the same out-degree.

In what follows we refer to this as to the constant out-degree condition.

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A less obvious necessary condition is called aperiodicity or primitivity:
the g.c.d. of lengths of all cycles should be equal to 1.

To see why primitivity is necessary, suppose that $\Gamma = (V, E)$ is a strongly connected digraph and $k > 1$ is a common divisor of lengths of its cycles. Take a vertex $v_0 \in V$ and, for $i = 0, 1, \ldots, k-1$, let

$$V_i = \{v \in V \mid \exists \text{ path from } v_0 \text{ to } v \text{ of length } i \pmod{k}\}.$$

Clearly, $V = \bigcup_{i=0}^{k-1} V_i$. We claim that $V_i \cap V_j = \varnothing$ if $i \neq j$.

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Let $v \in V_i \cap V_j$ where $i \neq j$. This means that in Γ there are two paths from v_0 to v: of length $\ell \equiv i \pmod{k}$ and of length $m \equiv j \pmod{k}$.

There is also a path from v to v_0 of length, say, n. Combining it with the two paths above we get a cycle of length $\ell + n$ and a cycle of length $m + n$.

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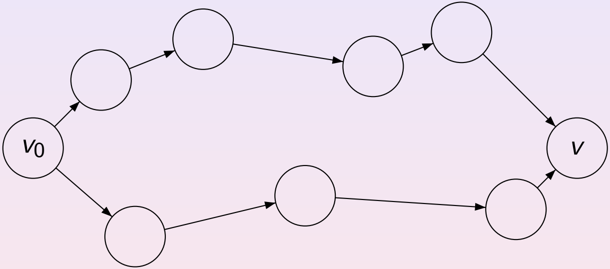

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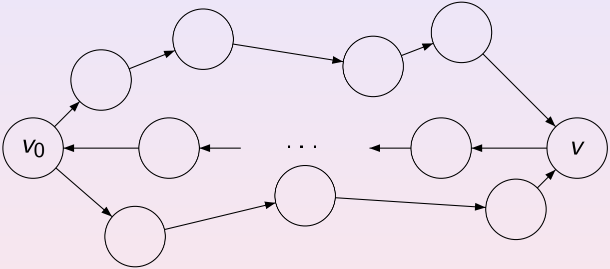

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Since k divides the length of any cycle in Γ, we have $\ell + n \equiv i + n \equiv 0 \pmod{k}$ and $m + n \equiv j + n \equiv 0 \pmod{k}$, whence $i \equiv j \pmod{k}$, a contradiction.

Thus, V is a disjoint union of $V_0, V_1, \ldots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Then Γ definitely cannot be converted into a synchronizing automaton by any labelling of its arrows: for instance, no paths of the same length ℓ originated in V_0 and V_1 can terminate in the same vertex because they end in $V_{\ell \pmod{k}}$ and in $V_{\ell+1 \pmod{k}}$ respectively.

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The Road Coloring Conjecture claims that the two necessary conditions (constant out-degree and primitivity) are in fact sufficient. In other words: *every strongly connected primitive digraph with constant out-degree admits a synchronizing coloring*.

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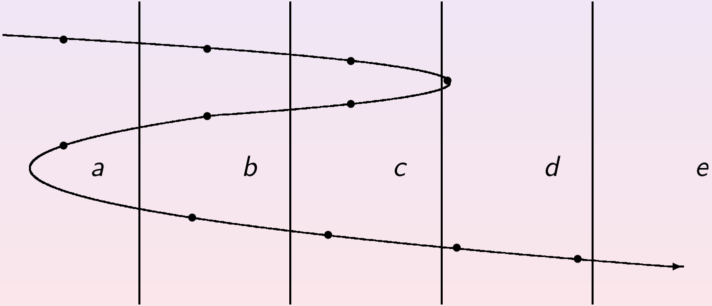

A symbolic trajectory $\ldots abcdcbabcdd\ldots$

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$$q \sim q' \Longleftrightarrow \forall u \in \Sigma^* \; \exists v \in \Sigma^* \; q\,.\,uv = q'.uv\,.$$

$\sim$ is called the *stability relation* and any pair (q, q') such that $q \sim q'$ is called *stable*. It is immediate that $\sim$ is a congruence of the automaton $\mathscr{A}$. Also observe that $\mathscr{A}$ is synchronizing iff all pairs are stable.

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We say that a coloring of a digraph with constant out-degree is *stable* if the resulting automaton contains at least one stable pair (q, q') with $q \neq q'$. The crucial observation by Culik, Karhumäki and Kari is

Proposition CKK. *Suppose every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Then the Road Coloring Conjecture holds true.*

The proof is rather straightforward: one inducts on the number of vertices in the digraph. If Γ admits a stable coloring and $\mathscr{A}$ is the resulting automaton, then the quotient automaton $\mathscr{A}/{\sim}$ admits a synchronizing recoloring by the induction assumption.
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Example

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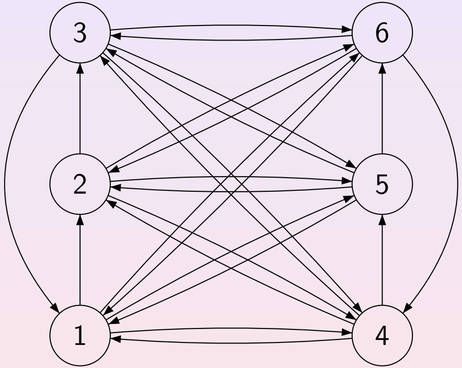

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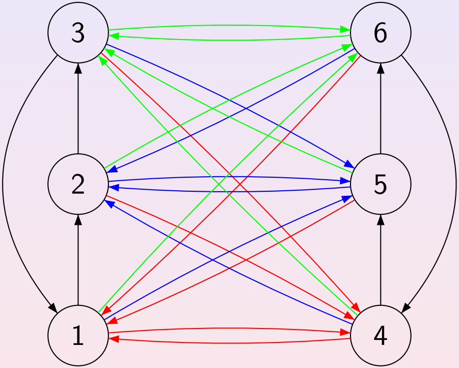

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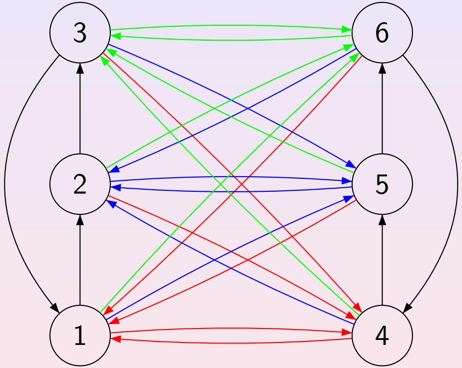

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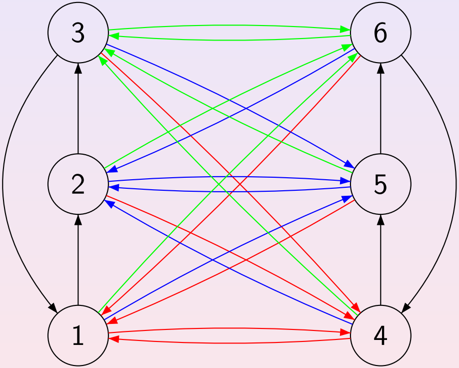

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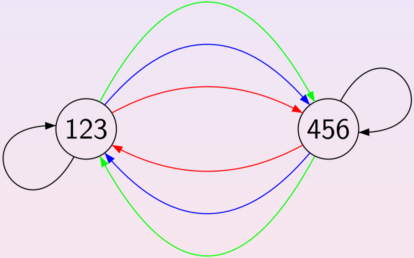

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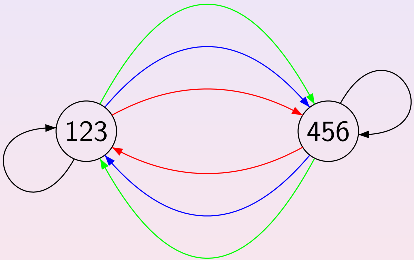

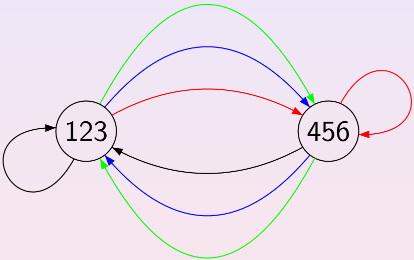

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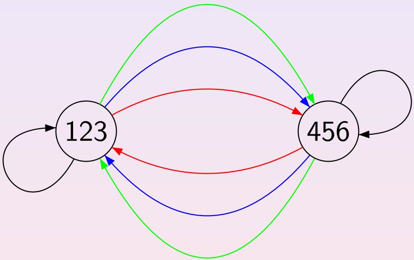

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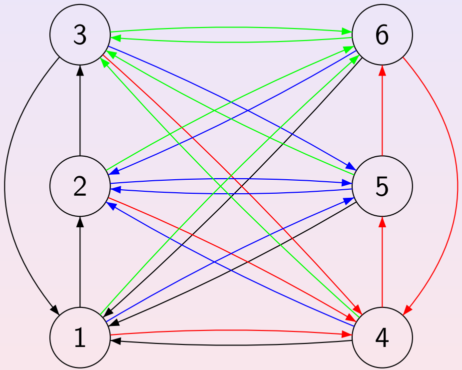

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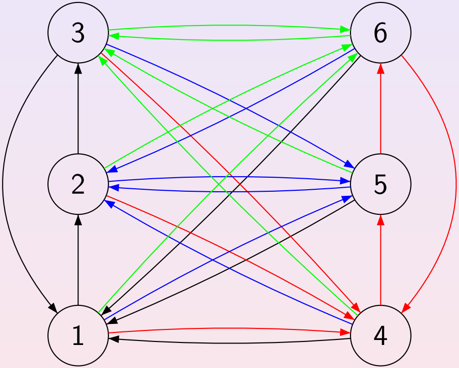

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Trahtman has managed to prove exactly what was needed to use Proposition CKK: every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Thus, Road Coloring Conjecture holds true.

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Deadlocks and Cliques

First, we need a couple of notions.

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. A pair (p, q) of distinct states is a deadlock if $\forall w \in \Sigma^* \; p \,.\, w \neq q \,.\, w$. If an automaton is not synchronizing, it must have deadlocks!

Moreover, if a pair (p, q) is not stable, then for some word $u \in \Sigma^*$ the pair $(p \,.\, u, q \,.\, u)$ is a deadlock.

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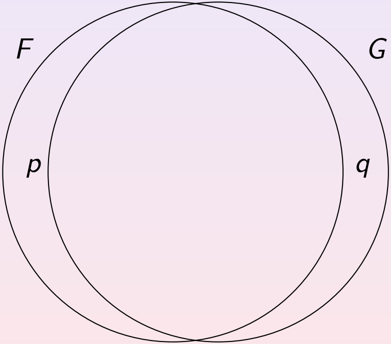

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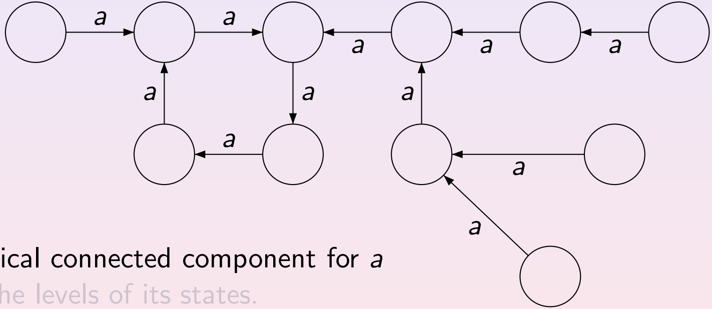

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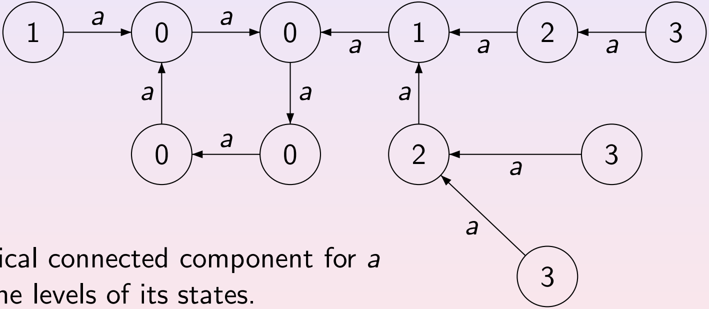

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Proof. Let M be the set of all states of level L w.r.t a. Then $p\,.\,a^L = q\,.\,a^L$ for all $p, q \in M$ whence no pair of states from M forms a deadlock. Thus, if $C \subseteq Q$ is a clique then $|C \cap M| \leq 1$. Take a clique C such that $|C \cap M| = 1$ (it exists since $\mathscr{A}$ is strongly connected). Then $F = C.a^{L-1}$ is a clique that has all its states except one in the a-cycles. If m is the l.c.m. of the lengths of all a-cycles, $r\,.\,a^m = r$ for any r in any a-cycle. Hence $G = F\,.\,a^m$ is a clique such that

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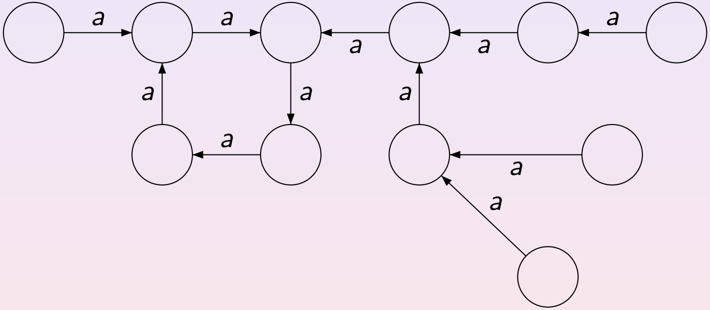

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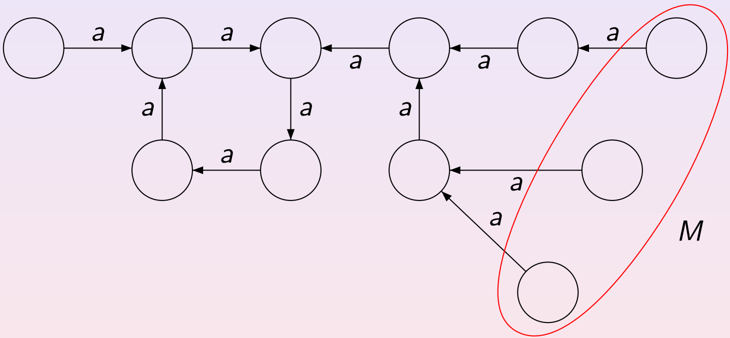

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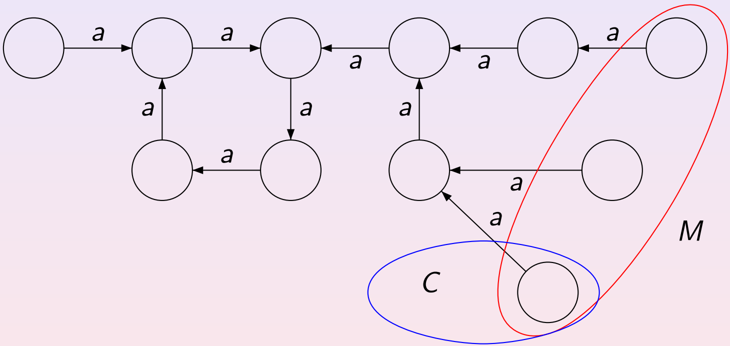

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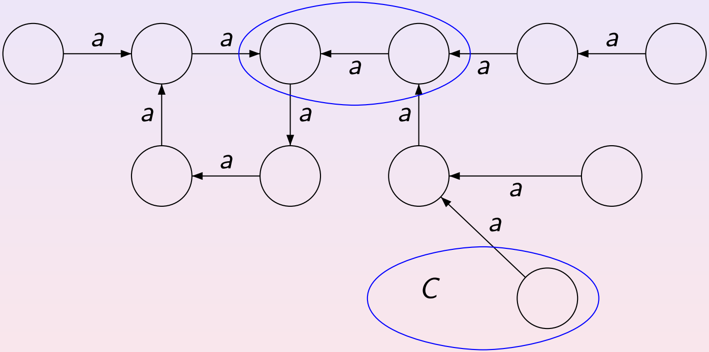

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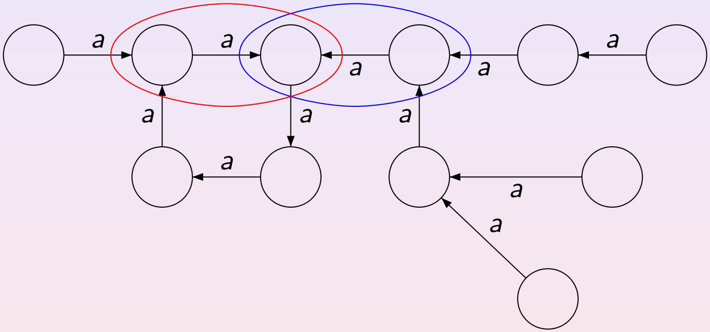

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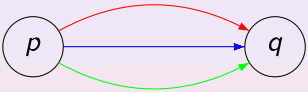

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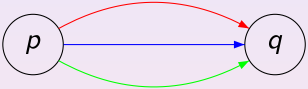

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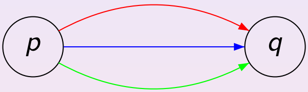

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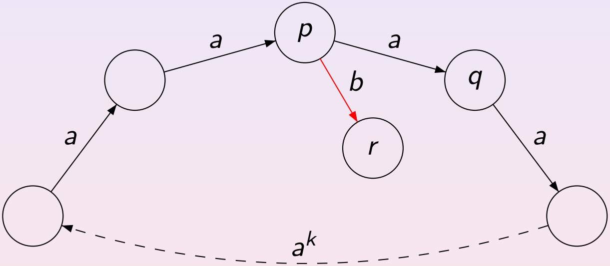
Thus, let p be a state which is not a bunch, let $q = p \, . \, a$ and let $b \neq a$ be such that $r = p \, . \, b \neq q$. We exchange the labels of the edges $p \xrightarrow{a} q$ and $p \xrightarrow{b} r$.



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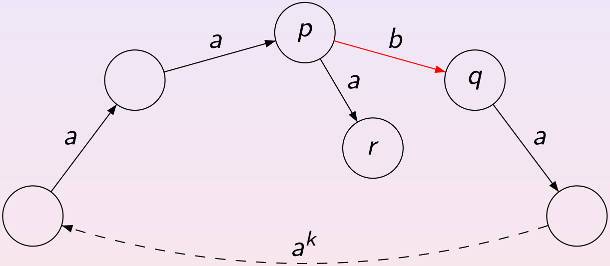

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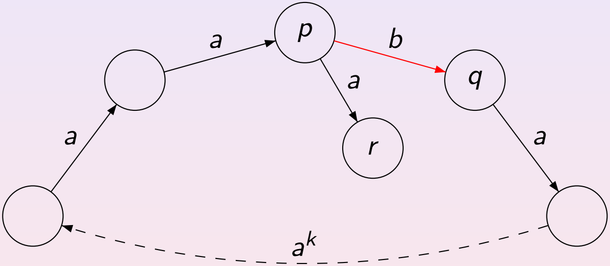

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Now let $N > 0$. We denote by L the maximum level of the states w.r.t. a in the initial coloring. Observe that $N > 0$ implies $L > 0$. Let p be a state of level L. Since Γ is strongly connected, there is an edge $p' \to p$ with $p' \neq p$, and by the choice of p, the label of this edge is $b \neq a$. Let $t = p' \,.\, a$. One has $t \neq p$. Let $r = p \,.\, a^L$ and let C be the a-cycle on which r lies.

The following considerations split in several cases. In each case except one we can recolor Γ by swapping the labels of two edges such the new coloring either satisfies the premise of Lemma 2 (all states of maximal level w.r.t. a belong to the same tree) or has more states on the a-cycles (and the induction assumption applies). The remaining case turns out to be easy.

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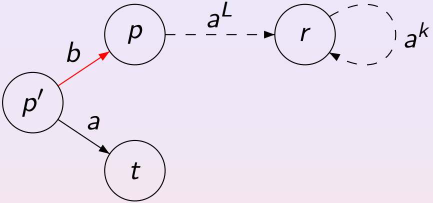

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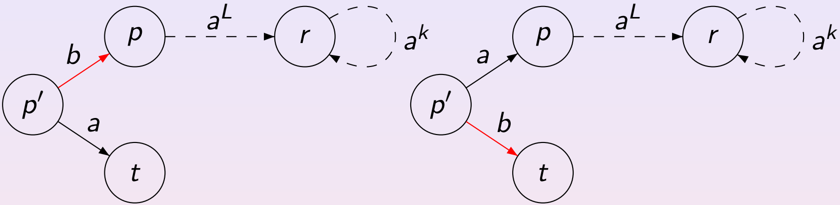

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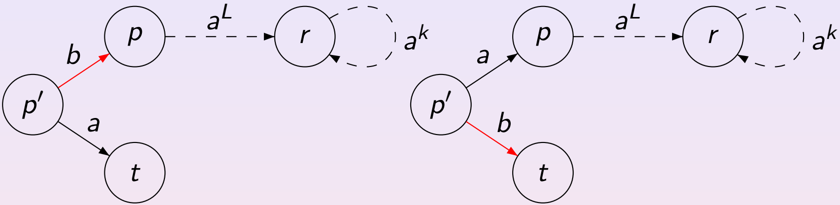

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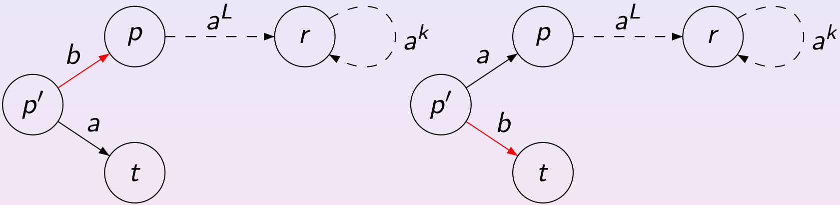

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Subcase 2.1: $k_2 \neq L$. Again, we swap the labels of $p' \xrightarrow{b} p$ and $p' \xrightarrow{a} t$. If $k_2 < L$, then the swapping creates an a-cycle of length $k_1 + L + 1 > k_1 + k_2 + 1$ increasing the number of states on the a-cycles. If $k_2 > L$, then the level of t w.r.t. a becomes k_2 whence all states of maximal level w.r.t. a in the new automaton are a-ascendants of t and thus belong to the same tree.

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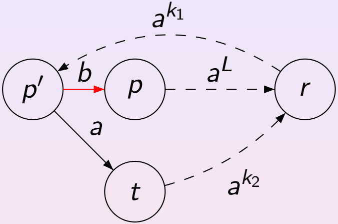

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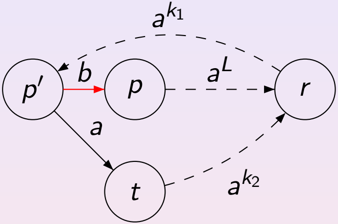

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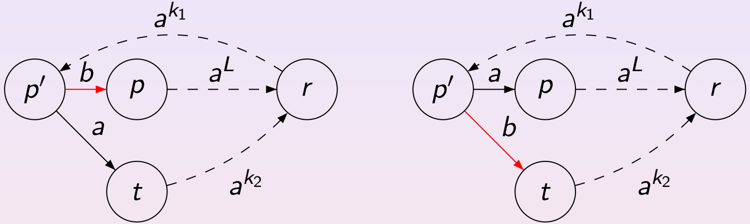

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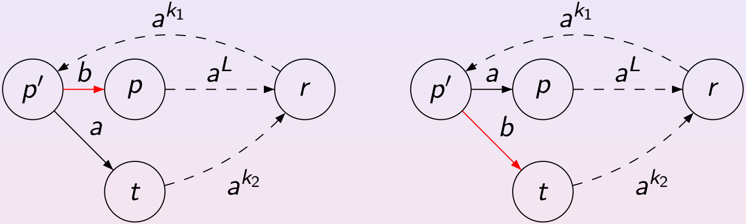

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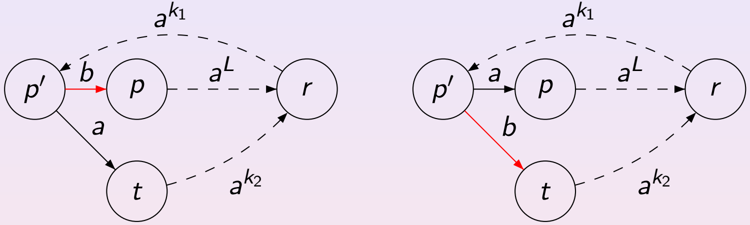

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Subcase 2.2: $k_2 = L$ and s is not a bunch. Since s is not a bunch, there is a letter c such that $s' = s\,.\,c \neq r$.

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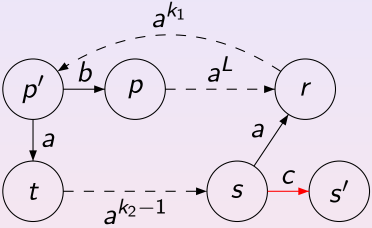

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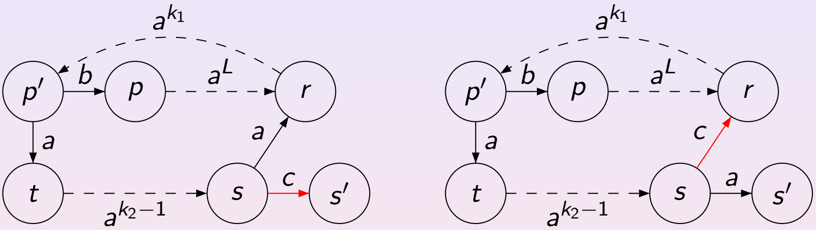

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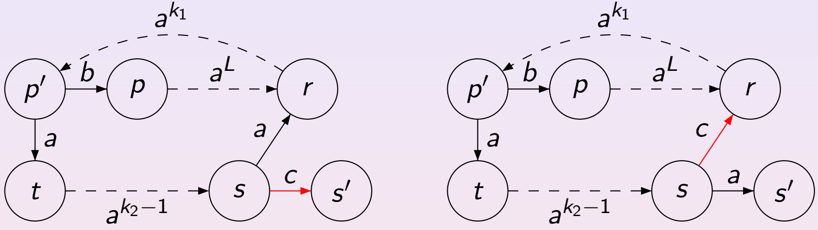

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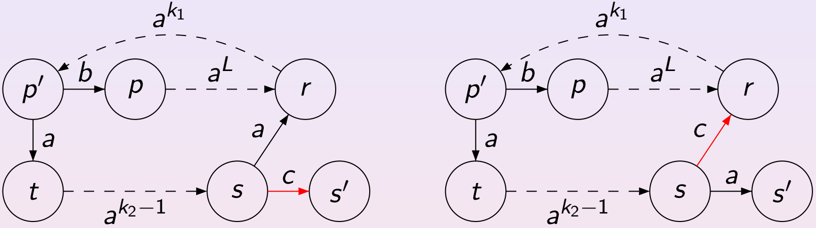

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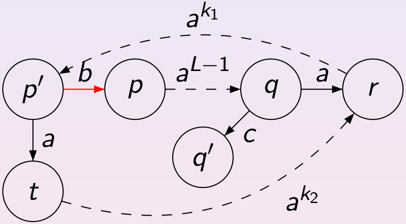

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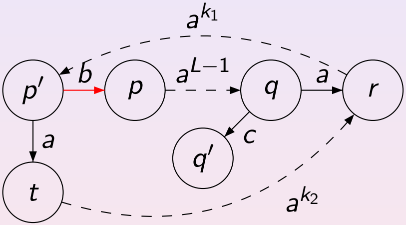

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Subcase 2.4: $k_2 = L$ and both s and q are bunches.

In this case it is clear that q and s form a stable pair. This completes the proof.

The proof can be 'unfolded' to a quadratic (in $|V|$) algorithm to find a synchronizing coloring of a given admissible digraph $\Gamma = (V, E)$ – Marie-Pierre Béal and Dominique Perrin, 2008.

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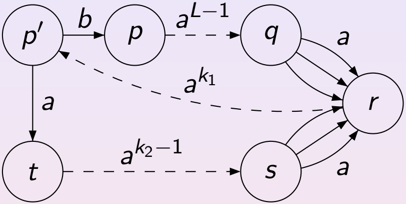

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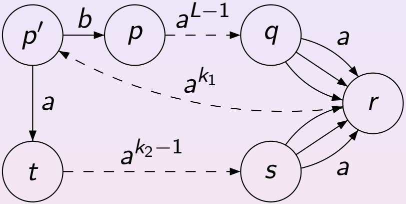

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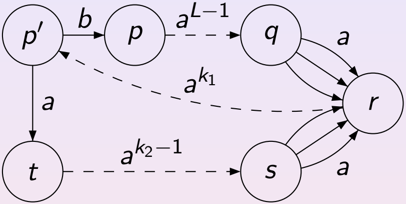

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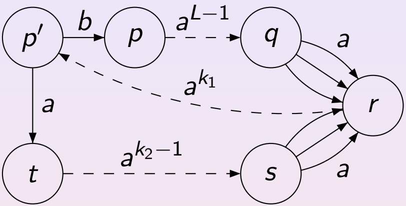

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An Application

Recall the connection between maximal prefix codes and automata discussed in Lecture 1.

Simple cycles in the automaton on the right correspond to codewords. Thus, if a finite maximal prefix code is such that the g.c.d. of lengths of its codewords is 1, then there exists a synchronized code with the same lengths of codewords (equivalently, with the same tree.) This was proved by Dominique Perrin and Marcel-Paul Schützenberger in 1992.

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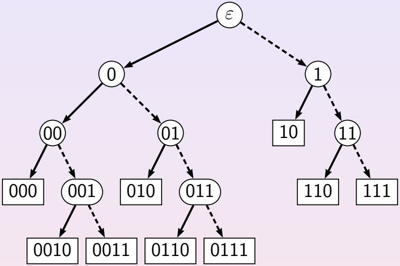

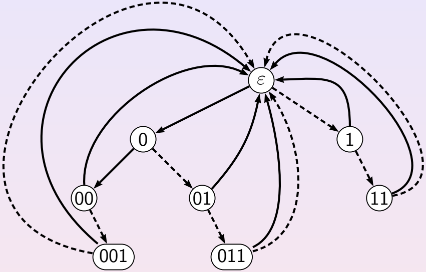

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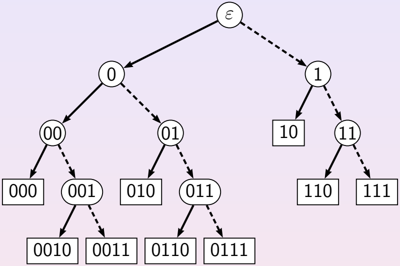

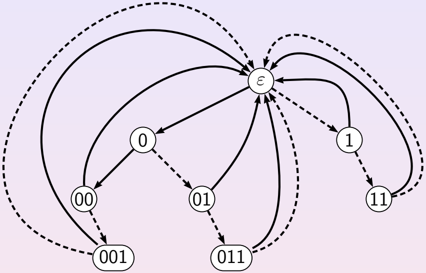

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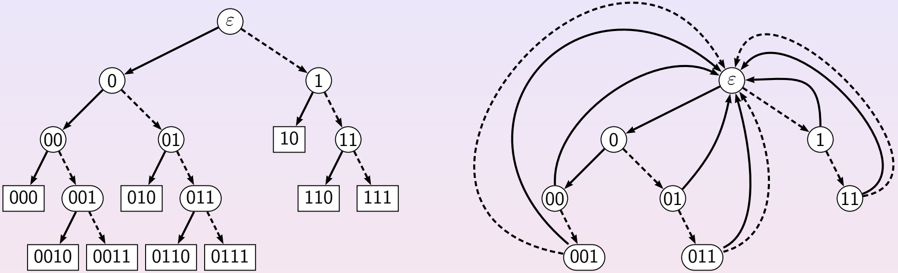

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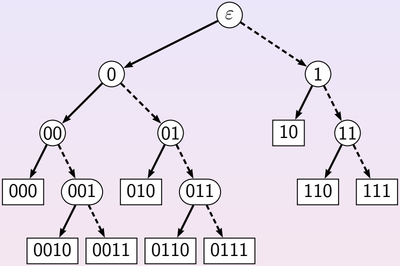

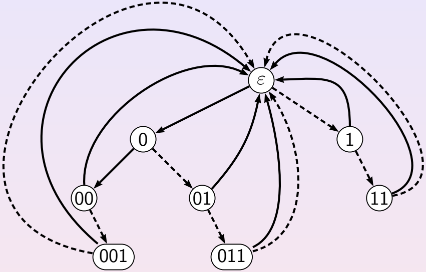

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