Character theory and abstract structure of finite groups

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This talk is a short survey of some results from the character theory of finite groups which are used for the study of the abstract structure of groups. In particular, some results of the author are discussed. We consider the following themes.

1. Some notation and elementary definitions.
2. Character table of a group.
3. Interactions and D-blocks.
4. Zeroes in the character table.
5. Characterization of groups by active fragments of the character table.
6. Semiproportional characters.

1. Further, G is a finite group and $\mathbb{C}$ is the field of all complex numbers. If $g \in G$ then $C_G(g)$ is the centralizer of g in G, $g^G := \{x^{-1}gx \mid x \in G\}$ is the conjugacy class of G containing g, and $k(G)$ is the number of conjugacy classes of G. We remember concepts: a *representation of G over a field F*; the *degree* of a representation; the *character* of a representation; *reducible* and *irreducible* representations; the *kernel* of a representation. The writing $D \trianglelefteq G$ denotes that D is a normal subset of G (i. e. the union of some conjugacy classes of G). Majority of necessary to us concepts and results may be find in [1–3].

2. A *character* (*irreducible character*) *of* G is a character of some representation (respectively, irreducible representation) of G over $\mathbb{C}$. Irr(G) denotes the set of all irreducible characters of G. Then $|\mathrm{Irr}(G)| = k(G)$. If $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_k\}$ and $\mathrm{Cl}(G) = \{{g_1}^G, {g_2}^G, \dots, {g_k}^G\}$, where $k = k(G)$, then ($k \times k$-matrix) $\mathrm{X}(G) = (\chi_i(g_j))$ ($k \times k$-matrix) is a *character table* of G (X is the Greek Chi). The *orthogonality relations* in $\mathrm{X}(G)$ are significant.

Problem 1. *To investigate the interdependency of the properties of the character table of a group and the abstract structure of this group.*

2a. From G to $\mathrm{X}(G)$: For any given group G may be constructed $\mathrm{X}(G)$ (see [2, theorem 10]).

2b. From $\mathrm{X}(G)$ to G: The size of the table $\mathrm{X}(G)$ is very small with respect to $|G|$ (examples are given) in order that determinate G (up to isomorphism) by $\mathrm{X}(G)$. $X(D_8) = X(Q_8)$ almost $D_8 \not\cong Q_8$. Nevertheless, it is possible recognize many properties of G from $\mathrm{X}(G)$. There exist some groups that may be completely reconstructed (up to isomorphism) by their character tables. In particular, this property have groups S_n [4] and A_n [5].

3. We remind the concepts of *interaction* and *D-block* (where $D \trianglelefteq G$) introduced in [6] (see also [3, chapter 3, sect. A]) and discuss some appropriate results of the author (in particular, [10]). The concept of D-block generalizes the classical concept of p-block (where p is a prime number): *If D is the set $G_{p'}$ of all p'-elements of G, then the concept of D-block coincides with the concept of p-block.* We discuss an effective method (from [7]) for calculating p-blocks of finite groups which is based on using of D-blocks for some p-sections D of a given group.

4. For applications of the character theory to study the abstract structure of groups, results on existence and disposition of zeros in $\mathrm{X}(G)$ are important. Here we give some examples of such results. In particular, one of such results (see [8]) has Corollary: if $\mathrm{X}(G)$ has a zero submatrix $O_{s\times t}$ then $s + t \leq k(G) - 1$. A zero submatrix $O_{s\times t}$ of $\mathrm{X}(G)$ with $s + t = k(G) - 1$ is called the *extremal zero fragment* of $\mathrm{X}(G)$. The following problem is not solved till now.

Problem 2. *To investigate groups G such that* $\mathrm{X}(G)$ *has an extremal zero fragment.*

5. Let $D \trianglelefteq G$, $\Phi \subseteq \mathrm{Irr}(G)$ and $\mathrm{X}(\Phi, D)$ is the submatrix of $\mathrm{X}(G)$ lying on intersections of rows corresponding to characters in Φ and columns corresponding to classes in D. If D is interact with Φ then the matrix $\mathrm{X}(\Phi, D)$ is called an *active fragment* of $\mathrm{X}(G)$ or an active fragment of G. We shall discuss some established by the author characterizations of finite groups (in particular, J_1 [9], $PSL_2(q)$ and $Sz(q)$) by their active fragments.

6. Functions φ and ψ from a set M in the field $\mathbb{C}$ is called *semiproportional*, if they are not proportional and there is a subset H in M such that $\varphi|_M$ is proportional to $\psi|_M$ and $\varphi|_{S\setminus M}$ is proportional to $\psi|_{S\setminus M}$ ($\varphi|_M$ denotes the restriction of φ to M). In particular, we may speak on semiproportional characters of a group, on semiproportional rows and on semiproportional column of $X(G)$. For brevity, two conjugacy classes of G we shall call *semiproportional* if corresponding to them columns of $X(G)$ are semiproportional. We discuss (see, in particular, [11–15]) the following conjectures.

Conjecture 1 (Semiproportional Characters Conjecture). If φ and ψ are semiproportional irreducible characters of a finite group then $\varphi(1) = \psi(1)$.

Conjecture 2 (Semiproportional Classes Conjecture). If g^G and h^G are semiproportional conjugacy classes of a finite group G then the cardinality of one from this classes divides the cardinality of other.

We discuss also some results connected with following theorem (the concluding result is obtained in [13]).

Theorem. *Finite alternating groups have no semiproportional irreducible characters.*

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