STUDY ABOUT THE ADJACENCY FOR THE DIFFERENT CLASSES OF THE BINARY RELATIONS

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1. Adjacency of binary relations

Definition 1. Let $B = \{0,1\}$ — Boolean set, X — arbitrary set, and $X^2 \doteq X \times X$ — a direct product. The functions $X^2 \to B$ will be called *characteristic*. Any subset $\sigma \subseteq X^2$, called a binary relation (or relation) on the set X, generates characteristic function

$$\chi_R\colon X^2 \to B, \qquad \chi_R(x,y) \doteq \begin{cases} 1, & \text{if } (x,y) \in R, \\ 0, & \text{if } (x,y) \notin R. \end{cases}$$

Next, the function $\chi_R(\cdot,\cdot)$ will be denoted by $R(\cdot,\cdot)$. On the other hand, any characteristic function $\chi\colon X^2 \to B$ generates a binary relation $R_\chi \subseteq X^2$ such that $(x,y) \in R_\chi$, if $\chi(x,y) = 1$. Obviously, the map $R \to R(\cdot,\cdot)$ is a bijection between the set of binary relations and the set of characteristic functions.

On the set 2^{X^2} of all binary relations of the set X we introduce a binary reflexive relation of adjacency.

Definition 2. Let $X = Y \cup Z$ — the disjoint union of two subsets (allowed that either $Y = \varnothing$ or $Z = \varnothing$). Suppose that the relation $\sigma \subseteq X^2$ such that $\sigma(x, y) = 0$ for all $(x, y) \in Y \times Z$. It generates the relation $\tau \subseteq X^2$ such that

$$\tau(x, y) = 1 - \sigma(y, x) \text{ для всех } (x, y) \in Y \times Z,$$

$$\tau(x, y) = 0 \text{ для всех } (x, y) \in Z \times Y,$$

$$\tau(x, y) = \sigma(x, y) \text{ для всех } (x, y) \in Y^2 \cup Z^2.$$

The relation τ is called *adjacent* with the relation σ.

Remark 1. From the definition it follows that if the relation τ adjacent with a relation σ, then σ adjacent with a relation τ, and this fact we write in the form of a diagram $\sigma \stackrel{Y\times Z}{\longleftrightarrow} \tau$ or

$$\begin{array}{c|c|c|} & Y & Z \\ \hline Y & & 0 \\ \hline Z & \sigma(x,y) & \\ \hline \end{array} = \sigma \stackrel{Y\times Z}{\longleftrightarrow} \tau = \begin{array}{|c|c|c} Y & Z & \\ \hline & 1-\sigma(y,x) & Y \\ \hline 0 & & Z \\ \hline \end{array}$$

Here and elsewhere in the diagrams we mark for the value of the characteristic functions at those points which are known a priori. For example, in the block $Y \times Z$ for the relation σ we write «generalized» zero, and this means that

$$\sigma(x, y) = 0 \quad \text{for all} \quad (x, y) \in Y \times Z,$$

and in the same block for the relation τ we write $1 - \sigma(y, x)$, and this means that

$$\tau(x, y) = 1 - \sigma(y, x) \quad \text{for all} \quad (x, y) \in Y \times Z.$$

For example, $X = \{1, \ldots, 6\}$, $Y = \{1, 2\}$, $Z = \{3, 4, 5, 6\}$,

$$\begin{array}{|c|c|}\hline \begin{matrix}1&0\\0&1\end{matrix} & \Large 0 \\ \hline \begin{matrix}0&0\\1&0\\0&0\\1&1\end{matrix} & \begin{matrix}1&0&1&0\\1&1&1&0\\0&0&1&0\\0&0&0&1\end{matrix} \\ \hline \end{array} \quad \overset{Y \times Z}{\longleftrightarrow} \quad \begin{array}{|c|c|}\hline \begin{matrix}1&0\\0&1\end{matrix} & \begin{matrix}1&0&1&0\\1&1&1&0\end{matrix} \\ \hline \Large 0 & \begin{matrix}1&0&1&0\\1&1&1&0\\0&0&1&0\\0&0&0&1\end{matrix} \\ \hline \end{array}$$

2. Adjacency of the partial orders

Let $V(X)$ is the collection of all partial orders defined on the set X. In other words, the relation $\sigma \subseteq X^2$ belongs to the set $V(X)$, if it satisfies the following axioms: 1) $(x,x) \in \sigma$ (reflexivity); 2) if $(x,y) \in \sigma$, $(y,z) \in \sigma$, then $(x,z) \in \sigma$ (transitivity); 3) if $(x,y) \in \sigma$, $(y,x) \in \sigma$, then $x = y$ (antisymmetry). In the terms of the characteristic we have: $\sigma \in V(X)$ *if and only if*

$$\sigma(x,x) = 1 \quad for\, all \quad x \in X,$$

$$\sigma(x,y)\,\sigma(y,z) \leqslant \sigma(x,z) \quad for\, all \quad x,y,z \in X,$$

$$\sigma(x,y)\,\sigma(y,x) = \delta_{xy} \quad for\, all \quad x,y \in X \;(where\; \delta_{xy} - Kronecker\; symbol).$$

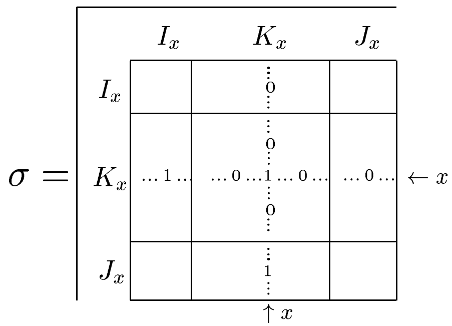
Theorem 1. (See [2].) *Let* σ *and* τ *— are adjacent relations. Inclusion* $\sigma \in V(X)$ *hold if and only if* $\tau \in V(X)$.

Thus, the set X generates a pair $\langle V(X), E(X) \rangle$, where $V(X)$ is a set of vertices, consisting of all partial orders and $E(X)$ is a set of edges, consisting of all unordered distinct pairs of adjacent partial orders of the set X. The pair $G(X) \doteq \langle V(X), E(X) \rangle$ will be called (undirected) graph of partial orders of the set X.

We say that the partial orders σ and τ belong to the same connected component of the graph $G(X)$, if there is a finite sequence of partial orders $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_m = \tau$, such that relations σ_{k-1} and σ_k are adjacent for all $k = 2, ..., m$. Let $G_\sigma(X)$ is the connected component of the graph $G(X)$ that contains the partial order σ.

3. On the features of the structure of the graph of partial orders

We fix the partial order $\sigma \in V(X)$ and an element $x \in X$. For σ we have the representation:



$$I_x \doteq I_x(\sigma) \doteq \{\, y \in X : \ \sigma(x,y) = 1,\ \sigma(y,x) = 0 \,\},$$
$$K_x \doteq K_x(\sigma) \doteq \{\, y \in X : \ \sigma(x,y) = \sigma(y,x) = \delta_{xy} \,\},$$
$$J_x \doteq J_x(\sigma) \doteq \{\, y \in X : \ \sigma(x,y) = 0,\ \sigma(y,x) = 1 \,\}.$$

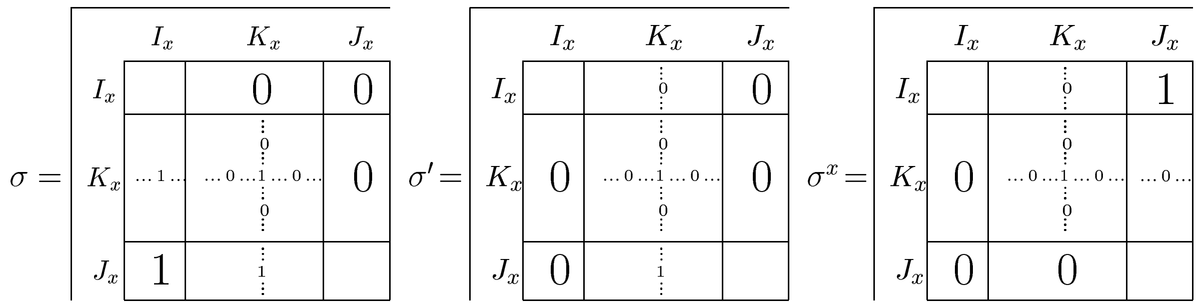
Obviously , $x \in K_x$.

Lemma 1. *The following equalities holds:*

1) $\sigma(y,z) = 1$ *for all* $(y,z) \in J_x \times I_x$,
2) $\sigma(y,z) = 0$ *for all* $(y,z) \in I_x \times (K_x \cup J_x)$,
3) $\sigma(y,z) = 0$ *for all* $(y,z) \in (I_x \cup K_x) \times J_x$.

Hence we can construct a sequence of adjacent partial orders

$$\sigma \xleftrightarrow{I_x \times (K_x \cup J_x)} \sigma' \xleftrightarrow{(I_x \cup K_x) \times J_x} \sigma^x, \tag{1}$$



which leads us to the partial order $\sigma^x \in V(X)$, that $\sigma^x(x,y) = \sigma^x(y,x) = \delta_{xy}$ for all $y \in X$ (in other words, if we interpret the partial order as relation $\leqslant$, then x it is both a maximum and minimum element of a partial order σ^x).

Thus for a fixed partial order $\sigma \in V(X)$ are defined a map $X \longrightarrow G_\sigma(X)$, associated to an element $x \in X$ the partial order $\sigma^x \in G_\sigma(X)$ (it may be that $\sigma^x = \sigma^y$ at least $x \neq y$). We also note that this map is uniquely defined — in the algorithm (1) we use uniquely defined sets $I_x(\sigma)$, $K_x(\sigma)$, $J_x(\sigma)$.

Lemma 2. *Let the partial orders $\sigma, \tau \in V(X)$ belong to the same connected component of the graph $G(X)$. Then $\sigma^x = \tau^x$ for any $x \in X$.*

Corollary 1. In connected component of the graph $G_\sigma(X)$ for each $x \in X$ there is only partial order τ such that $\tau(x,y) = \tau(y,x) = \delta_{xy}$ at all $y \in X$, and $\tau = \sigma^x$.

If $\operatorname{card} \mathrm{X} < \infty$ (we can assume that $X = \{1, \ldots, n\}$ is the segment of the natural numbers), then there is a one-to-one correspondence between the set $V(X)$ and the set of all labeled transitive graphs defined on X in its turn there is a one-to-one correspondence between this set and the set of all labeled T_0-topologies, defined on X (we denote the number of these topologies by $T_0(n)$).

Theorem 2. (See [2].) *If $n \geq 2$ and $X = \{1, \ldots, n\}$ then* $\operatorname{card} V(X) = T_0(n)$, *and the number of the connected component of the graph $G(X)$ equal to $T_0(n-1)$.*

4. Support sets of the partial orders

The set $S(\sigma) \doteq \{\, y \in X : \ \sigma(x,y) = \delta_{xy}$ for all $x \in X \,\}$ is called *support set* (or *support*) of the partial order $\sigma \in V(X)$. If $S(\sigma) = X$ then σ is called *trivially* (or *discrete*) partial order.

Proposition 1. *If* $\operatorname{card} X < \infty$ *then* $S(\sigma) \neq \varnothing$ *for any* $\sigma \in V(X)$.

Lemma 3. *Let* $\operatorname{card} X < \infty$. *For any nontrivial partial order* $\sigma \in V(X)$ *and for any* $y \in X \backslash S(\sigma)$ *there is* $x \in S(\sigma)$ *such that* $\sigma(x,y) = 1$.

Corollary 2. *Let* $\operatorname{card} X < \infty$ *and* $\sigma \in V(X)$ *such that* $S(\sigma) = \{x\}$ *– singleton. Then* $\sigma(x,y) = 1$ *for any* $y \in X$.

Lemma 4. *Let* $\operatorname{card} X < \infty$ *and the partial orders* σ *and* τ *belong to the same of the connected component of the graph* $G(X)$. *Then* $S(\sigma) = S(\tau)$ *if and only if* $\sigma = \tau$.

Proposition 2. *Let* $\operatorname{card} X < \infty$. *For any partial order* $\sigma \in V(X)$ *and for any nonempty subset* $S \subseteq S(\sigma)$ *there is only partial order* τ, *belonging to the connected component* $G_\sigma(X)$ *of the graph* $G(X)$, *such that* $S(\tau) = S$.

Proposition 3. *Let* $\operatorname{card} X < \infty$. *In any connected component* $G_\sigma(X)$ *of the graph* $G(X)$ *and for any nonempty subset* $S \subseteq X$, *consisting of not more than two elements, there is only partial order* τ *such that* $S(\tau) = S$.

Remark 2. Let $S(G_\sigma) \doteq \{ S(\tau) \subseteq X : \tau \in G_\sigma(X) \}$ — the collection of all support sets of the partial orders which belong in the component $G_\sigma(X)$.

By propositions 1–3 the following assertions hold:

1) $\varnothing \notin S(G_\sigma)$;

2) if $\varnothing \neq \alpha \subseteq \beta \subseteq X$ and $\beta \in S(G_\sigma)$ then $\alpha \in S(G_\sigma)$;

3) if $\alpha \subseteq X$ and $|\alpha| \leq 2$ then $\alpha \in S(G_\sigma)$.

In other words, a set $S(G_\sigma)$ is a partially ordered set with respect to natural relation of inclusion of sets with the following specifics:

1) with every element the set $S(G_\sigma)$ contains all its non-empty subsets;

2) $S(G_\sigma)$ contains all one- and two-element subsets of X.

Thus, to describe a partially ordered set $S(G_\sigma)$ is sufficient to indicate all his maximums.

Remark 3. Let $\mathrm{card} X = n$. By theorem 2 and proposition 3 it follows that the set $V(X)$ has exactly:

1) $n\, T_0(n-1)$ different partial orders, support set which contains exactly one element;

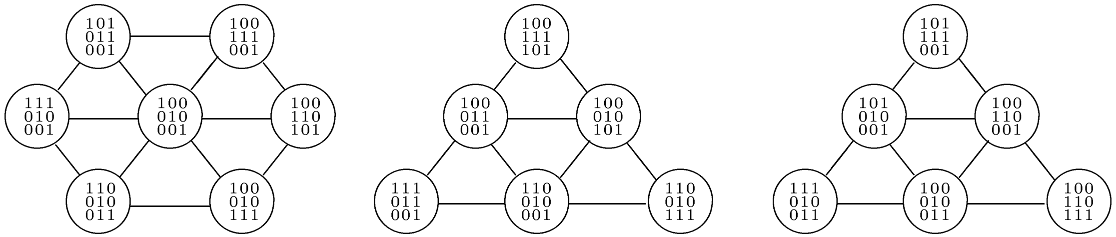
2) $\frac{1}{2}\, n\,(n-1)\, T_0(n-1)$ different partial orders, support set which contains exactly two elements.

Hence, take place a theorem below previously proven in independent works[1].

Theorem. *For any* $n \geq 2$ *the following equality holds:*

$$T_0(n) = \tfrac{1}{2}\, n\,(n+1)\, T_0(n-1) + \operatorname{card}\big\{\, \sigma \in V(\{1,\dots,n\}) \colon\ |S(\sigma)| \geq 3 \,\big\}.$$

Examples. We present below 3 components of the graph $G(\{1,2,3\})$ containing 19 partial orders (it is well known that $T_0(2) = 3$, a $T_0(3) = 19$):



We denote the components of the graphs by K_1, K_2 and K_3. Obviously, the components K_2 and K_3 isomorphic (if to apply, for example, substitution $\binom{123}{213}$ to the elements of the components K_2, we obtain K_3). The set $S(K_1)$ consists of a single maximum element $\{1,2,3\}$ and all of its non-empty subsets, and the set $S(K_2)$ contains three maximum elements: $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. In the graph there is only one partial order, which $|S(\sigma)| \geq 3$, — trivial.

[1] **1) Родионов В.И.** Об одном соотношении в конечных топологиях // Зап. научн. сем. ЛОМИ. 1980. Т. 103. С. 114–116. **2) Erne M.** On the cardinalities of finite topologies and the number of antichains in partially ordered sets, *Discrete Mathematics*, 1981, vol. 35, pp. 119–133.

The graph $G(\{1,2,3,4\})$ consists of 19 connected components and contains 219 vertices (we know values $T_0(n)$ for all $n \leqslant 12$, in particular, $T_0(4) = 219$). We presented below vertices of three connected components of this graph (K_1, K_2 and K_3 respectively):

1111	1000	1011	1000	1101	1000	1001	1000	1110	1000	1010	1000	1100	1000	1000
0100	1111	0111	0100	0100	1101	0101	0100	0100	1110	0110	0100	0100	1100	0100
0010	0010	0010	1111	0111	1011	0011	0010	0010	0010	0010	1110	0110	1010	0010
0001	0001	0001	0001	0001	0001	0001	1111	0111	1011	0011	1101	0101	1001	0001

1111	1000	1011	1000	1100	1000	1000	1010	1110	1000	1010	1000
0100	1111	0111	0100	0100	1100	0100	0110	0100	1110	0110	0100
0011	0011	0011	1111	0111	1011	0011	0010	0010	0010	0010	0010
0001	0001	0001	1101	0101	1001	0001	1111	0101	1001	0001	1101

1111	1000	1000	1100	1100	1000	1110	1110	1000	1100
0111	1111	0111	0100	0100	0100	0110	0110	0110	0100
0011	1011	0011	1111	0011	1011	0010	0010	0010	0010
0001	1001	0001	1101	0001	1001	1111	0001	1001	1101

We have $|K_1| = 15$, $|K_2| = 12$, $|K_3| = 10$, an order of the automorphism group of component K_1 is 24, $|\mathrm{Aut}(K_2)| = 2$, $|\mathrm{Aut}(K_3)| = 4$, therefore 12 components of the graph isomorphic component K_2, and another 6 components of the graph isomorphic component K_3. This means that $219 = 1\cdot 15 + 12\cdot 12 + 6\cdot 10$. In the set $S(K_1)$ there is a unique maximal element $\{1,2,3,4\}$, in the set $S(K_2)$ there is three maximal elements: $\{1,2,3\}$, $\{1,2,4\}$ and $\{3,4\}$, in the set $S(K_3)$ there is six maximal elements: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ and $\{3,4\}$.

The graph $G(\{1,\ldots,5\})$ consists of 219 components and contains 4231 vertex. Below we present the representatives (one from each component, such that $S(\sigma)=\{1\}$) of seven connected components $K_1,\ldots,K_7$ of the graph, as well as the cards and orders of automorphism groups of components:

$i=$	1	2	3	4	5	6	7
	11111 01000 00100 00010 00001 $\vdots$	11111 01000 00100 00011 00001 $\vdots$	11111 01000 00101 00011 00001 $\vdots$	11111 01011 00101 00010 00001 $\vdots$	11111 01011 00101 00011 00001 $\vdots$	11111 01000 00111 00011 00001 $\vdots$	11111 01111 00111 00011 00001 $\vdots$
$\lvert K_i\rvert=$	31	24	22	20	19	18	15
$\lvert \mathrm{Aut}(K_i)\rvert=$	120	6	4	5	2	2	5

Thus, $T_0(5)=4231=\sum_{i=1}^{7}\frac{5!}{|\mathrm{Aut}(K_i)|}|K_i|$, $T_0(4)=219=\sum_{i=1}^{7}\frac{5!}{|\mathrm{Aut}(K_i)|}$. We list the maximal element of $S(K_i)$. In the set $S(K_1)$ — the maximal element is $\{1,\ldots,5\}$, in the set $S(K_2)$ — three maximal elements: $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{4,5\}$, in the set $S(K_3)$ — also three maximal elements: $\{1,2,3,4\}$, $\{1,2,5\}$, $\{3,4,5\}$, in the set $S(K_4)$ — five maximal elements: $\{1,2,3\}$, $\{1,3,4\}$, $\{1,4,5\}$, $\{2,3,5\}$, $\{2,4,5\}$, in the set $S(K_5)$ — six maximal elements: $\{1,2,3\}$, $\{1,3,4\}$, $\{1,5\}$, $\{2,3,5\}$, $\{2,4\}$, $\{3,4,5\}$, in the set $S(K_6)$ — also six maximal elements: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{3,4\}$, $\{3,5\}$, $\{4,5\}$, and finally, in the set $S(K_7)$ there are ten maximal elements of the form $\{k,m\}$, $k\neq m$.

As are known all the maximal element of the set $S(K_i)$, then the card $|K_i| = |S(K_i)|$ can be calculated in accordance with the principle of inclusion-exclusion. For example,

$$|K_3| = |S(K_3)| = \left[\,2^{|\{1,2,3,4\}|}-1\,\right] + \left[\,2^{|\{1,2,5\}|}-1\,\right] + \left[\,2^{|\{3,4,5\}|}-1\,\right] - \\ -\left[\,2^{|\{1,2\}|}-1\,\right] - \left[\,2^{|\{3,4\}|}-1\,\right] - \left[\,2^{|\{5\}|}-1\,\right] = 15+7+7-3-3-1 = 22.$$

The graph $G(\{1,\ldots,6\})$ consists of 4231 components and contains 130023 vertex. The following are the cards and the orders of groups of automorphisms of the graph components $K_1,\ldots,K_{18}$:

$\|K_i\|=$	63	48	42	42	37	36	36	34	33	33	32	30	30	29	29	27	25	21
$\|\mathrm{Aut}(K_i)\|=$	720	24	12	12	4	2	24	6	2	2	2	8	4	1	6	1	2	6

and the following equalities hold:

$$T_0(6) = 130023 = \sum_{i=1}^{18} \frac{6!}{|\mathrm{Aut}(K_i)|}\,|K_i|, \qquad T_0(5) = 4231 = \sum_{i=1}^{18} \frac{6!}{|\mathrm{Aut}(K_i)|}\,.$$

5. Reflexive-transitive relations

Let $V(X)$ is the collection of all reflexive-transitive relations defined on the set X. In the other words, the relation $\sigma \subseteq X^2$ belong in $V(X)$, if it satisfies axioms reflexivity $\big((x,x)\in\sigma\big)$ and transitivity $\big($ if $(x,y)\in\sigma$, $(y,z)\in\sigma$, then $(x,z)\in\sigma\big)$. In terms of characteristic functions we have: $\sigma \in V(X)$ *if and only if*

$$\sigma(x,x) = 1 \quad \textit{for all} \quad x \in X,$$

$$\sigma(x,y)\,\sigma(y,z) \leqslant \sigma(x,z) \quad \textit{for all} \quad x,y,z \in X.$$

For any $\sigma \in V(X)$ and $x \in X$ the set $U_\sigma(x) \doteq \big\{\, y \in X:\ \sigma(x,y) = 1 \,\big\}$ is not empty $\big($since $x \in U_\sigma(x)\big)$.

Proposition 4. *Let $\sigma \in V(X)$ and $x,y \in X$. Then $y \in U_\sigma(x)$ if and only if $U_\sigma(y) \subseteq U_\sigma(x)$.*

The relation $\sigma \in V(X)$ generates an equivalence relation on the set X: write $x \sim y$ (or $x \overset{\sigma}{\sim} y$) if $U_\sigma(x) = U_\sigma(y)$. The equivalence class containing the element $x \in X$ denote by $[x]_\sigma$ (or $\overline{x}$).

Proposition 5. *Let $\sigma \in V(X)$ and $x, y \in X$. The following assertions hold:*
1) $[x]_\sigma \subseteq U_\sigma(x)$;
2) *if $y \in U_\sigma(x)$ then $[y]_\sigma \subseteq U_\sigma(x)$; therefore* $U_\sigma(x) = \bigcup\limits_{[\xi]_\sigma \subseteq U_\sigma(x)} [\xi]_\sigma$;
3) $\sigma(\xi, \eta) = 1$ *for all* $(\xi, \eta) \in [x]_\sigma^2$;
4) $\sigma(\xi, \eta) = \sigma(x, y)$ *for all* $(\xi, \eta) \in [x]_\sigma \times [y]_\sigma$;
5) *if* $[x]_\sigma \neq [y]_\sigma$ *then* $\sigma(\xi, \eta)\, \sigma(\eta, \xi) = 0$ *for all* $(\xi, \eta) \in [x]_\sigma \times [y]_\sigma$.

6. Graph of reflexive-transitive relations

Theorem 3. (See [4].) *Let* σ *and* τ *— are adjacent relations defined on the set* X*. Then* $\sigma \in V(X)$ *if and only if* $\tau \in V(X)$*.*

Thus, the set X generates a pair $\big\langle V(X), E(X)\big\rangle$, where $V(X)$ this is a set of vertices consisting of all reflexive-transitive relations and $E(X)$ — this is a set of edges, consisting of all unordered distinct pairs of adjacent reflexive-transitive relations of the set X. The pair $G(X) \doteq \big\langle V(X), E(X)\big\rangle$ will be called (undirected) graph of reflexive-transitive relations of the set X.

We say that the reflexive-transitive relations σ and τ belong in the same connected component of the graph $G(X)$, if there is a finite sequence of reflexive-transitive relations $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_m = \tau$, such that the relations σ_{k-1} and σ_k are adjacent for all $k = 2, ..., m$. Through $G_\sigma(X)$ we denote the connected component of the graph $G(X)$, which contains the reflexive-transitive relation σ.

7. Features of the structure of the graph reflexive-transitive relations

Let $\sigma \in V(X)$. Through $[X]_\sigma$ denoted the set of all equivalence classes of the set X (i.e. $[X]_\sigma = \big\{ [x]_\sigma \big\}_{x\in X} = \big\{\overline{x}\big\}_{x\in X}$). Due to the point 4 of proposition 5 we can define the following characteristic function $\bar\sigma\colon\ [X]_\sigma^2 \to B$ such that

$$\bar\sigma\,(\bar x,\bar y) \doteq \sigma(\xi,\eta),$$

where (ξ,η) is any pair in the direct product $\bar x \times \bar y$.

It is clear that

$$\bar\sigma\,(\bar x,\bar x) = \sigma(x,x) = 1 \quad \mathit{for\ all} \quad \bar x \in [X]_\sigma;$$

$$\bar\sigma\,(\bar x,\bar y)\,\bar\sigma\,(\bar y,\bar z) = \sigma(x,y)\,\sigma(y,z) \leqslant \sigma(x,z) = \bar\sigma\,(\bar x,\bar z) \quad \mathit{for\ all} \quad \bar x,\bar y,\bar z \in [X]_\sigma;$$

$$\bar\sigma\,(\bar x,\bar y)\,\bar\sigma\,(\bar y,\bar x) = \sigma(x,y)\,\sigma(y,x) = \delta_{\bar x\,\bar y} \quad \mathit{for\ all} \quad \bar x,\bar y \in [X]_\sigma,$$

$$\mathit{where}\ \delta_{\bar x\,\bar y} - \mathit{symbol\,Kronecker}.$$

This means, that σ generates a partial order $\bar\sigma$ on the set $[X]_\sigma$. Consequently, in accordance with the concept of graph of the partial orders that σ generates a graph $G_0([X]_\sigma) \doteq \big\langle V_0([X]_\sigma), E([X]_\sigma)\big\rangle$, where $V_0([X]_\sigma)$ — this is a set of partial orders defined on the set $[X]_\sigma$, and $E([X]_\sigma)$ — this is a set of edges, consisting of unordered pairs of distinct adjacent partial orders of the set $[X]_\sigma$.

Thus, $\bar\sigma \in V_0([X]_\sigma)$ and $\bar\sigma$ determines the connected component of the graph $G_0([X]_\sigma)$, containing a partial order $\bar\sigma$ $\big($denoted by $G_0^{\bar\sigma}([X]_\sigma)\big)$.

Example 1. Let $X = \{1,2,3\}$, $\sigma = \left[\begin{smallmatrix}1&1&1\\1&1&1\\0&0&1\end{smallmatrix}\right]$. Then $G_\sigma(X) = \left\langle \textcircled{\begin{smallmatrix}1&1&1\\1&1&1\\0&0&1\end{smallmatrix}} - \textcircled{\begin{smallmatrix}1&1&0\\1&1&0\\0&0&1\end{smallmatrix}} - \textcircled{\begin{smallmatrix}1&1&0\\1&1&0\\1&1&1\end{smallmatrix}} \right\rangle$,

$$[X]_\sigma = \{\overline{1}, \overline{3}\} = \{\{1,2\},\{3\}\}, \quad \overline{\sigma} = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right], \quad G_0([X]_\sigma) = G_0^{\overline{\sigma}}([X]_\sigma) = \left\langle \textcircled{\begin{smallmatrix}1&1\\0&1\end{smallmatrix}} - \textcircled{\begin{smallmatrix}1&0\\0&1\end{smallmatrix}} - \textcircled{\begin{smallmatrix}1&0\\1&1\end{smallmatrix}} \right\rangle.$$

We say that the relations $\sigma, \tau \in V(X)$ is reflexive-transitive relations of the same type if $[X]_\sigma = [X]_\tau$.

Proposition 6. *If σ and τ — are adjacent reflexive-transitive relations defined on the set X, then $[X]_\sigma = [X]_\tau$ (that is both are of the same type).*

Remark 4. In the proof of the above proposition we have shown in particular that if $\sigma \xleftrightarrow{Y \times Z} \tau$ then for any $x \in X = Y \cup Z$ takes place an alternative: *either* $\bar{x} \subseteq Y$ *or* $\bar{x} \subseteq Z$. In the other words, the set $[X] \doteq [X]_\sigma = [X]_\tau$ can be represented as a disjoint union

$$[X] = [Y] \cup [Z], \text{ where } [Y] \doteq \{\bar{x} \in [X] \colon \bar{x} \subseteq Y\}, \quad [Z] \doteq \{\bar{x} \in [X] \colon \bar{x} \subseteq Z\}.$$

Remark 5. As a result, we found that every relation $\sigma \in V(X)$ generates a connected component $G_\sigma(X)$ of the graph $G(X)$, the set $[X]_\sigma$ of equivalence classes, partial order $\bar{\sigma} \in V_0([X]_\sigma)$, the graph $G_0([X]_\sigma)$ and his connected component $G_0^{\bar{\sigma}}([X]_\sigma)$. Furthermore, if $\tau \in G_\sigma(X)$ then $G_\tau(X) = G_\sigma(X)$, $[X]_\tau = [X]_\sigma$, and the following proposition is proved this equality $G_0^{\bar{\tau}}([X]_\tau) = G_0^{\bar{\sigma}}([X]_\sigma)$.

Proposition 7. *Let σ and τ are reflexive-transitive relations defined on the set X, and $\bar{\sigma}$, $\bar{\tau}$ — is generated by them partial orders defined on the sets $[X]_\sigma$ and $[X]_\tau$ respectively. Relations σ and τ are adjacent if and only if the relations $\bar{\sigma}$ and $\bar{\tau}$ are adjacent relations.*

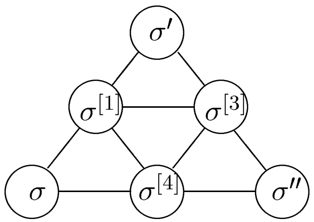
Proposition 8. *Let $\sigma \in V(X)$. Connected graphs $G_\sigma(X)$ and $G_0^{\bar{\sigma}}([X]_\sigma)$ are isomorphic.*

Example 2. In the example 1 graphs $G_\sigma(X)$ and $G_0^{\bar{\sigma}}([X]_\sigma)$ are isomorphic and

$$\sigma^{[1]} = \sigma^{[2]} = \sigma^{[3]} = \begin{bmatrix} 1&1&0 \\ 1&1&0 \\ 0&0&1 \end{bmatrix}.$$

If $X = \{1, 2, 3, 4\}$, $\sigma = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{bmatrix}$, then $\sigma^{[1]} = \sigma^{[2]} = \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{bmatrix}$, $\sigma^{[3]} = \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&0 \\ 1&1&0&1 \end{bmatrix}$, $\sigma^{[4]} = \begin{bmatrix} 1&1&1&0 \\ 1&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$,

$G_\sigma(X)$:



$$\sigma' = \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 1&1&1&1 \\ 1&1&0&1 \end{bmatrix}, \quad \sigma'' = \begin{bmatrix} 1&1&1&0 \\ 1&1&1&0 \\ 0&0&1&0 \\ 1&1&1&1 \end{bmatrix}.$$

8. Bijection between finite reflexive-transitive relations and finite topologies

The relation $\sigma \in V(X)$ will be called *finite*, if the set $[X]_\sigma$ consists of a finite number of equivalence classes, i.e. card $[\mathrm{X}]_\sigma < \infty$. The collection of all these relations denote by $W(X)$. Obviously, $W(X) \subseteq V(X)$.

We fix an arbitrary set X and topology T on X.

Next, we suppose that T is the finite topology on the set X. Then for any $x \in X$ there is the smallest open set $S_{\mathrm{T}}(x)$, containing the point x, where $S_{\mathrm{T}}(x)$ is the intersection of all open sets containing the point x.

Proposition 9. *Let* T *is finite topology defined on the set* X, $S \in \mathrm{T}$, $x, y, z \in X$.
1. *If* $x \in S$ *then* $S_{\mathrm{T}}(x) \subseteq S$ *and* $S = \bigcup_{x \in S} S_{\mathrm{T}}(x)$.
2. $y \in S_{\mathrm{T}}(x)$ *if and only if* $S_{\mathrm{T}}(y) \subseteq S_{\mathrm{T}}(x)$.
3. *If* $y \in S_{\mathrm{T}}(x)$, $z \in S_{\mathrm{T}}(y)$ *then* $z \in S_{\mathrm{T}}(x)$.

Let $T(X)$ is the collection of all finite topologies defined on set X.

Proposition 10. *The mapping* $\Phi\colon T(X) \to W(X)$ *is bijective.*

9. Graph of finite topologies

Let $\operatorname{card} \mathrm{X} < \infty$ (we can assume that $X = \{\, 1, \ldots, n \,\}$ is a segment of the natural numbers). Obviously, any topology T, is defined on X, that is $\mathrm{T} \in T(X)$. For any $\sigma \in V(X)$ we have $\operatorname{card} [\mathrm{X}]_\sigma < \infty$, therefore $\sigma \in W(X)$. Hence, $W(X) = V(X)$ и $\operatorname{Im} \Phi = V(X)$.

By virtue of the bijection $\Phi^{-1} \colon V(X) \to T(X)$ we can assume that the vertices of the graph $\big\langle\, V(X), E(X) \big\rangle$ is finite topologies (elements of the set $T(X)$). It is possible to say that the topologies $\mathrm{T}, \mathrm{T}' \in T(X)$ are *adjacent*, if $\Phi(\mathrm{T}), \Phi(\mathrm{T}') \in V(X)$ are adjacent relations. You can also say that $\big\langle\, T(X), E(X) \big\rangle$ — *graph of finite topologies*.

In the example 2 the topology $\Phi^{-1}(\sigma) = \big\{\, \varnothing, \{1,2,3,4\}, \{3,4\}, \{4\} \,\big\}$ is the adjacent with the topologies

$$\Phi^{-1}(\sigma^{[1]}) = \big\{\, \varnothing, \{1,2\}, \{3,4\}, \{4\}, \{1,2,4\}, \{1,2,3,4\} \,\big\},$$

$$\Phi^{-1}(\sigma^{[4]}) = \big\{\, \varnothing, \{1,2,3\}, \{3\}, \{4\}, \{3,4\}, \{1,2,3,4\} \,\big\},$$

which, in turn, also adjacent.

The family $\{X_1, \ldots, X_m\}$, consisting of the subsets of the set X, we will call its *partition*, if $\bigcup\limits_{k=1}^{m} X_k = X$ and $X_i \cap X_j = \varnothing$ if $i \neq j$. (Obviously $m \leq n$.) The set of all partitions of X will be denoted by $\mathcal{P}(X)$, and $\mathcal{P}_m(X)$ will be denoted the family of all partitions which are have exactly m component. We have the equality $\operatorname{card} \mathcal{P}_m(X) = S(n, m)$, where $S(n, m)$ — Stirling numbers of the 2nd kind defined by well-known formula

$$S(n, m) = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n.$$

Obviously, for any $\sigma \in V(X)$ the family $[X]_\sigma$ is a partition of the set X.

Through $V_0(X)$ will be denoted the set of all partial orders defined on the set X. There is a one-to-one correspondence between the set $V_0(X)$ and the set of all labeled transitive graphs defined on X in turn, there is a one-to-one correspondence between this set and the set of labeled T_0-topologies defined on X, and $T_0(n)$ will be denote the number of these topologies . We suppose $T_0(0) = 1$.

Theorem 4. (See [4].) *Let* $n \in \mathbb{N}$, $G(X) \doteq \langle V(X), E(X) \rangle$ *— the graph of transitive-reflexive relations defined on the set* $X \doteq \{1, \ldots, n\}$. *Then*

$$\operatorname{card} V(X) = \sum_{m=1}^{n} S(n,m)\, T_0(m),$$

and if $[\,G(X)\,]$ *is the set of connected components of the graph* $G(X)$ *then*

$$\operatorname{card}[\,G(X)\,] = \sum_{m=1}^{n} S(n,m)\, T_0(m-1).$$

10. Adjacency of acyclic digraphs. Let $X \doteq \{1, \ldots, n\}$ and $A(X)$ is the collection of all labeled acyclic digraphs (acyclic relations) defined on X.

Theorem 5. (Unpublished.) *Let σ and τ — are adjacent relations. Inclusion $\sigma \in A(X)$ holds if and only if $\tau \in A(X)$.*

Thus, the set X generates a graph $\langle A(X), E(X) \rangle$, where $A(X)$ is a set of vertices, consisting of all labeled acyclic digraphs, defined on the set X, and $E(X)$ is a set of edges, consisting of all unordered pairs of various adjacent acyclic digraphs.

Theorem 6. (Unpublished.) *If $X \doteq \{1, \ldots, n\}$ then*[1]

$$\operatorname{card} A(X) = \sum_{p_1 + \ldots + p_k = n} (-1)^{n-k} \frac{n!}{p_1! \ldots p_k!} \, 2^{(n^2 - p_1^2 - \ldots - p_k^2)/2},$$

and the number of connected components of the graph $\langle A(X), E(X) \rangle$ is equal to

$$\sum_{p_1 + \ldots + p_k = n} \frac{(-1)^{n-k}}{k} \frac{n!}{p_1! \ldots p_k!} \, 2^{(n^2 - p_1^2 - \ldots - p_k^2)/2},$$

where the summation in the sums carried out over all ordered sets $(p_1, \ldots, p_k)$ of natural integers such that $p_1 + \ldots + p_k = n$.

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