On finite 5-primary groups G with disconnected Gruenberg — Kegel graph and restrictions on $\pi_1(G)$

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Basic definitions and notations

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G.

n-primary group

Group G is called *n-primary* if $|\pi(G)| = n$.

Prime graph

Prime graph (or *Gruenberg — Kegel graph*) $\Gamma(G)$ of G is defined as the graph with vertex set $\pi(G)$, in which two vertices p and q are adjacent if and only if G contains an element of order pq.

We denote the number of connected components of $\Gamma(G)$ by $s(G)$, and the set of its connected components by $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$; for the group G of even order believe that $2 \in \pi_1(G)$.

Basic definitions and notations

Socle of a group

Subgroup $Soc(G)$ of G, generated by all minimal normal subgroups of G, is called *socle* of G.

Almost simple group

Group G is called *almost simple*, if $P = Soc(G)$ is non-abelian simple group, i. e. $Inn(P) \cong P \leq G \cong H \leq Aut(P)$.

Gruenberg — Kegel theorem

If G is a finite group with disconnected prime graph, then one of the following statements holds:

- G is a Frobenius group;
- G is a 2-Frobenius group (i.e. there exist subgroups A, B and C of G, such as $G = ABC$, where A and AB are normal in G, AB and BC are Frobenius groups with kernel A and B and complement B and C respectively);
- G is an extension of a nilpotent $\pi_1(G)$-group by an almost simple group A with socle P, in addition, $s(G) \leq s(P)$ and A/P is a $\pi_1(G)$-group.

Formulation of the problem

Let G be a finite group with disconnected prime graph isomorphic neither to a Frobenius group nor to a 2-Frobenius group and $F(G) \neq 1$.

$\overline{G} := G/F(G)$ is almost simple and is known.

Any connected component $\pi_i(G)$ of the graph $\Gamma(G)$ for $i > 1$ corresponds to a nilpotent isolated $\pi_i(G)$-Hall subgroup $X_i(G)$ of G. Any non-trivial element x of $X_i(G)$ for $i > 1$ acts *freely* (*without fixed points*) on $F(G)$.

Let K and L be neighboring terms of a chief series of G ($K < L$) containing in $F(G)$. Then, the chief factor $V = L/K$ is an elementary abelian p-group for some prime p (we will call it a *p-chief factor* of G), and it can be regarded as a faithful irreducible $GF(p)\overline{G}$-module.

Brief background

- A. S. Kondrat'ev and I. V. Khramtsov studied the finite groups having disconnected prime graph with the number of vertices not greater than 4 [since 2010];
- A. S. Kondrat'ev determined finite almost simple 5-primary groups and their Gruenberg — Kegel graphs [2014];
- V. K. and A. S. Kondrat'ev obtained a description of chief factors of the commutator subgroups of finite non-solvable 5-primary groups G with disconnected Gruenberg-Kegel graph in the case when $G/F(G)$ is almost simple n-primary group for $n \leq 4$ [2015].

Theorem 1

Let G be a finite 5-primary group and $\pi_1(G) = \{2\}$. Then one of the following conditions holds:
(1) $G \cong O(G) \leftthreetimes S$ is Frobenius group, where $O(G)$ is 4-primary abelian group and S is a cyclic 2-group or generalized quaternion group;
(2) G is Frobenius group with kernel $O_2(G)$ and 4-primary complement;
(3) $G \cong A \leftthreetimes (B \leftthreetimes C)$ is 2-Frobenius group, where $A = O_2(G)$, B is a cyclic 4-primary $2'$-group and C is a cyclic 2-group;
(4) $G \cong L_2(r)$, $r \geq 65537$ is Mersenne or Ferma prime and $|\pi(r^2 - 1)| = 4$;
(5) $\overline{G} = G/O_2(G) \cong L_2(2^m)$, where either $m \in \{6, 8, 9\}$, or $m \geq 11$ is prime. If $O_2(G) \neq 1$, then $O_2(G)$ is a direct product of minimal normal subgroups of order 2^{2m} from G, each of these as $\overline{G}$-module is isomorphic to the natural $GF(2^m)SL_2(2^m)$-module;
(6) $\overline{G} = G/O_2(G) \cong Sz(q)$, where $q = 2^p$, $p \geq 7$ and $q - 1$ primes, $|\pi(q - \varepsilon\sqrt{2q} + 1)| = 2$ and $|\pi(q + \varepsilon\sqrt{2q} + 1)| = 1$ for $\varepsilon \in \{+, -\}$, $5 \in \pi(q - \varepsilon\sqrt{2q} + 1)$. If $O_2(G) \neq 1$, then $O_2(G)$ is a direct product of minimal normal subgroups of order q^4 from G, each of these as $\overline{G}$-module is isomorphic to the natural $GF(q)Sz(q)$-module of dimension 4.

Theorem 2

Let G be a finite 5-primary group with disconnected prime graph, $\overline{G} = G/F(G)$ is almost simple 5-primary group, $3 \in \pi(G)$ and $3 \notin \pi_1(G) \neq \{2\}$. Then one of the following conditions holds:
(1) G is isomorphic to $L_2(5^3)$ or $L_2(17^3)$;
(2) $G \cong L_2(p)$, where either $p \geq 65537$ is Mersenne or Ferma prime and $|\pi(p^2-1)| = 4$, or $p \geq 41$ is prime, $|\pi(p^2-1)| = 4$ and $3 \in \pi(\frac{p+1}{2})$;
(3) G is isomorphic to $L_2(3^r)$ or $PGL_2(3^r)$, where r is odd prime, $|\pi(3^{2r}-1)| = 4$ and $r \notin \pi(G)$;
(4) $G \cong L_2(p^r)$, where $p \in \{5, 17\}$, r is odd prime, $|\pi(p^{2r}-1)| = 4$, $3 \in \pi(\frac{p^r+1}{2})$ and $r \notin \pi(G)$.

Theorem [Higman (1968), Stewart (1973)]

Let G be a finite group, $1 \neq H \trianglelefteq G$, and $G/H \cong L_2(2^n)$, where $n \geq 2$. Suppose that $C_H(t) = 1$ for some element t of order 3 from G. Then $H = O_2(G)$ and H is the direct product of minimal normal subgroups of order 2^{2n} in G such that each of them as G/H-module isomorphic to the natural $GF(2^n)SL_2(2^n)$-module.

Proposition [Stewart (1973)]

Let G be a finite group, $H \trianglelefteq G$, $G/H \cong L_2(q)$, where q is odd, $q > 5$, and $C_H(t) = 1$ for some element t of order 3 from $G \setminus H$. Then $H = 1$.

Lemma

Let G be a finite simple group, F be a field of characteristic $p > 0$, V be an absolutely irreducible FG-module, and β be a Brauer character of the module V. If g is an element of G of prime order different from p, then

$$dim\, C_V(g) = (\beta|_{\langle g \rangle}, 1|_{\langle g \rangle}) = \frac{1}{|g|} \sum_{x \in \langle g \rangle} \beta(x).$$

Thank you for attention.