On non-bipartite distance-regular graphs with small smallest eigenvalue

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Outline

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Defintion

Graph: $\Gamma = (V, E)$ where V vertex set, $E \subseteq \binom{V}{2}$ edge set.

- All graphs in this talk are simple.
- $x \sim y$ if $xy \in E$.
- $x \not\sim y$ if $xy \notin E$.
- $d(x, y)$: length of a shortest path connecting x and y.
- $D(\Gamma)$ diameter (max distance in Γ)

Distance-regular graphs

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- A connected graph Γ is called **distance-regular** (DRG) if there are numbers a_i, b_i, c_i $(0 \leq i \leq D = D(\Gamma))$ s.t. if $d(x,y) = j$ then
  - $\#\Gamma_1(y) \cap \Gamma_{j-1}(x) = c_j$
  - $\#\Gamma_1(y) \cap \Gamma_j(x) = a_j$
  - $\#\Gamma_1(y) \cap \Gamma_{j+1}(x) = b_j$

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Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
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- $H(n, 2) = n$-cube.
- DRG with $c_i = i$.
- Gives an algebraic frame work to study codes, especially bounds on codes.
- For example the Delsarte linear programming bound and more recently the Schrijver bound.

Eigenvalues of graphs

- Let Γ be a graph.
- The **adjacency matrix** for Γ is the symmetric matrix A indexed by the vertices st. $A_{xy} = 1$ if $x \sim y$, and 0 otherwise.
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- As A is a real symmetric matrix all its eigenvalues are real. We mainly will look at the smallest eigenvalue.

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1. The odd polygons with valency 2;
2. The complete tripartite graphs $K_{t,t,t}$ with valency $2t$ at least 2;
3. The folded $(2D+1)$-cubes with valency $2D+1$ and diameter $D \geq 2$;
4. The Odd graphs with valency k at least 3;
5. The Hamming graphs $H(D,3)$ with valency $2D$ where $D \geq 2$;
6. The dual polar graphs of type $B_D(2)$ with $D \geq 2$;
7. The dual polar graphs of type ${}^2A_{2D-1}(2)$ with $D \geq 2$.

Conjecture

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If $D > 0$ is large enough, and the smallest eigenvalue is not larger than $-k/2$, then Γ is a member of one of the seven families.

Outline

1. Defintions
   - Distance-Regular Graphs
   - Examples
2. Smallest eigenvalue is not larger than $-k/2$
   - Examples
   - A Valency Bound
   - Diameter 2

Valency Bound

Theorem

For any real number $1 > \alpha > 0$ and any integer $D \geq 2$, the number of coconnected (i.e. the complement is connected) non-bipartite distance-regular graphs with valency k at least two and diameter D, having smallest eigenvalue $\theta_{\min}$ not larger than $-\alpha k$, is finite.

Remarks

- Note that the regular complete t-partite graphs $K_{t \times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.

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- Note that the regular complete t-partite graphs $K_{t\times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.
- Note that there are infinitely many bipartite distance-regular graphs with diameter 3, for example the point-block incidence graphs of a projective plane of order q, where q is a prime power.
- The second largest eigenvalue for a distance-regular graphs behaves quite differently from its smallest eigenvalue. For example $J(n, D)$ $n \geq 2D \geq 4$, has valency $D(n-D)$, and second largest eigenvalue $(n-D-1)(D-1)-1$. So for fixed diameter D, there are infinitely many Johnson graphs $J(n, D)$ with second largest eigenvalue larger then $k/2$.

Outline

Coconnected

Let Γ be a distance-regular graph with valency $k \geq 2$ and smallest eigenvalue $\lambda_{\min} \leq -k/2$. It is easy to see that if the graph is coconnected then $a_1 \leq 1$.

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Diameter 2

1. The pentagon with intersection array $\{2, 1; 1, 1\}$;
2. The Petersen graph with intersection array $\{3, 2; 1, 1\}$;
3. The folded 5-cube with intersection array $\{5, 4; 1, 2\}$;
4. The 3×3-grid with intersection array $\{4, 2; 1, 2\}$;
5. The generalized quadrangle $GQ(2, 2)$ with intersection array $\{6, 4; 1, 3\}$;
6. The generalized quadrangle $GQ(2, 4)$ with intersection array $\{10, 8; 1, 5\}$;
7. A complete tripartite graph $K_{t,t,t}$ with $t \geq 2$, with intersection array $\{2t, t-1; 1, 2t\}$;

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No suprises.

Diameter 3 and triangle-free

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Our result:

Diameter 3

1. The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;
2. The Odd graph with valency 4, O_4, with intersection array $\{4, 3, 3; 1, 1, 2\}$;
3. The Sylvester graph with intersection array $\{5, 4, 2; 1, 1, 4\}$;
4. The second subconstituent of the Hoffman-Singleton graph with intersection array $\{6, 5, 1; 1, 1, 6\}$;
5. The Perkel graph with intersection array $\{6, 5, 2; 1, 1, 3\}$;

Diameter 3 and triangle-free, II

Theorem continued

1. The folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$;
2. A possible distance-regular graph with intersection array $\{7, 6, 6; 1, 1, 2\}$;
3. A possible distance-regular graph with intersection array $\{8, 7, 5; 1, 1, 4\}$;
4. The truncated Witt graph associated with M_{23} with intersection array $\{15, 14, 12; 1, 1, 9\}$;
5. The coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$;

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So this means that for diameter 3 and triangle-free, we obtain quite a few more examples, then the members of the three families.

We obtained also a classification of diameter 3 and 4 for distance-regular graphs having a triangle and smallest eigenvalue at most $-k/2$.

Thank you for attention.