COMPACTIFICATIONS OF MODULI SCHEMES FOR STABLE VECTOR BUNDLES ON A SURFACE, BY LOCALLY FREE SHEAVES

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Actuality

Main motive is investigation of moduli for connections in vector bundles: the Kobayashi – Hitchin correspondence allows to apply algebro-geometric methods to the problems in differential geometry and gauge theory.

moduli of connections in a vector bundle

$$\Updownarrow$$

moduli of slope-stable vector bundles

Also tools and results of gauge theory and differential geometry become applicable in the context of algebraic geometry.

X – compact complex algebraic surface,
E – differentiable complex vector bundle on X,
Hermitian metrics g on X and h on E.
$M_g^{st}(E)$ – moduli space for isomorphism classes of g-stable holomorphic structures on E,
$M_g^{HE}(E,h)$ – moduli space of gauge equivalence classes of h-unitary g-Hermitian – Einsteinian connections in the vector bundle E.

There exist a bijection $M_g^{HE}(E,h) \to M_g^{st}(E)$ *inducing an isomorphism of real analytic structures on these moduli spaces.*

For rank-2-vector bundles J.Li showed that

the Donaldson – Uhlenbeck compactification of moduli space of (gauge equivalence classes of) anti-self-dual connections admits such a complex structure that there is an induced reduced projective scheme structure on this moduli space. In this case the Gieseker – Maruyama compactification for the moduli scheme of stable vector bundles has a morphism on the scheme of anti-self-dual connections.

Maruyama 1977-1978: Moduli schemes of stable vector bundles are usually not projective and noncompact. It is useful to include the moduli scheme (variety) of vector bundle as open subscheme in some appropriate projective scheme. This problem is called traditionally as a problem of **compactification** of moduli space.

The classical solution is
Gieseker – Maruyama compactification: coherent Gieseker-semistable torsion-free sheaves with same Hilbert polynomial on the same variety are considered. S-equivalence classes of such coherent sheaves with same numerical invariants on the same surface are attached in "limit points" of families of vector bundles.

To build up a compactification of moduli of vector bundles (=locally free sheaves) it is necessary to allow the degeneration to nonlocally free coherent torsion-free sheaves. Because of this degeneration, the Gieseker – Maruyama compactification is not always convenient.

Other compactifications

Compactifications concerning with Yang – Mills field theory:
Donaldson – Uhlenbeck (1988)
(so-called ideal connections involved) and
Taubes – Uhlenbeck – Feehan (1995),
and also
D. Markushevich, A. Tikhomirov and G. Trautmann (2012)
announced in general case and constructed for rank 2 case the algebro-geometric analog of Taubes – Uhlenbeck – Feehan compactification.
It involves vector bundles on surfaces of come special form obtained by blowups of the initial surface in sequences of reduced points. This compactification is complete algebraic space.

Further prospective

We hope on possibility to build up an analog of the construction we present here, in the category of complex analytic spaces, constructibility of Kobayashi – Hitchin correspondence on complex analytic spaces which correspond to admissible schemes.

This will enable us to investigate moduli of connections in terms of compactifications constructed.

Aim

to interpret the degeneration of semistable locally free sheaves on a surface in flat families in terms of degeneration of the surface as locally free sheaves degenerate in locally free sheaves.

Main content

Alternative compactifications of moduli of stable vector bundles on a nonsingular projective algebraic surface S over a field $k = \overline{k}$ of zero characteristic, are constructed. Such compactifications can me obtained when we allow degeneraion of the surface S in projective algebraic schemes of certain class as local freeness of sheaves is preserved.
The compactifications constructed are projective algebraic schemes.

We **choose and fix** a very ample invertible sheaf L on S.
Also once arbitrarily chosen and fixed are
rank $r = \operatorname{rank} E$ and
reduced Hilbert polynomial $p(n)$
for coherent sheaves on the surface S.
The symbol $\overline{M}$ means the moduli scheme of Gieseker-semistable torsion-free coherent sheaves on S, of rank r and reduced Hilbert polynomial compute w.r.t. L and equal to $p(n)$.

Definition[Gieseker, D., 1977] The coherent torsion-free $\mathcal{O}_S$-sheaf E is **Gieseker-stable** (resp., **Gieseker-semistable**) if for any subsheaf $F \subset E$ при $n \gg 0$

$$\frac{\chi(E \otimes L^n)}{\text{rank}\, E} > \text{(resp., } \geq) \frac{\chi(F \otimes L^n)}{\text{rank}\, F}$$

Three types of compactifications are built:

constructive $\widetilde{M}_c$,
reduced $\widetilde{M}_{\text{red}}$, and
nonreduced $\widetilde{M}$.

Types are inspired by the mode of construction.

Remarks, restrictions, conventions

S – smooth irreducible projective algebraic surface over alg. closed field k of zero characteristic.

Variety is a reduced separated Noetherian scheme of finite type over a field. Variety can be uniquely decomposed into the union of irreducible components. These components are integral separated schemes of finite type. By bijectivity of correspondense between vector bundles and locally free sheaves on the same algebraic scheme, both terms are used as synonyms. In the case of arbitrary algebraic scheme X, there is a maximal under inclusion reduced subscheme $X_{\text{red}} \subset X$. It is closed in X and is defined by the nilradical $\mathcal{N}il(\mathcal{O}_X)$ – the sheaf of ideals which is generated by nilpotent elements in $\mathcal{O}_X$. Such a subscheme is called a *reduction* of the scheme X. The corresponding subscheme in the moduli scheme is called a *reduction of moduli scheme* or *reduced moduli scheme*. Moduli schemes under consideration are Noetherian schemes of finite type. When being separated, reduced moduli schemes correspond to algebraic varieties, and we call them *moduli varieties*.

Notations

$\Sigma_{\mathrm{red}} := \overline{M}_{\mathrm{red}} \times S$,
$\Sigma_{\mathrm{red0}} := M_{\mathrm{red0}} \times S$, where
M_{red0} is open subscheme in $\overline{M}_{\mathrm{red}}$ whose points correspond to stable locally free sheaves,
M'_{red0} is open subscheme in $\overline{M}_{\mathrm{red}}$ whose points correspond to semistable locally free sheaves,
$p : \Sigma_{\mathrm{red}} \to \overline{M}_{\mathrm{red}}$ is a projection on the first factor.

Resolution of a family of semistable coherent sheaves on a surface S into the family of locally free sheaves on the family of schemes ("modilied surfaces") of certain form, is constructed.

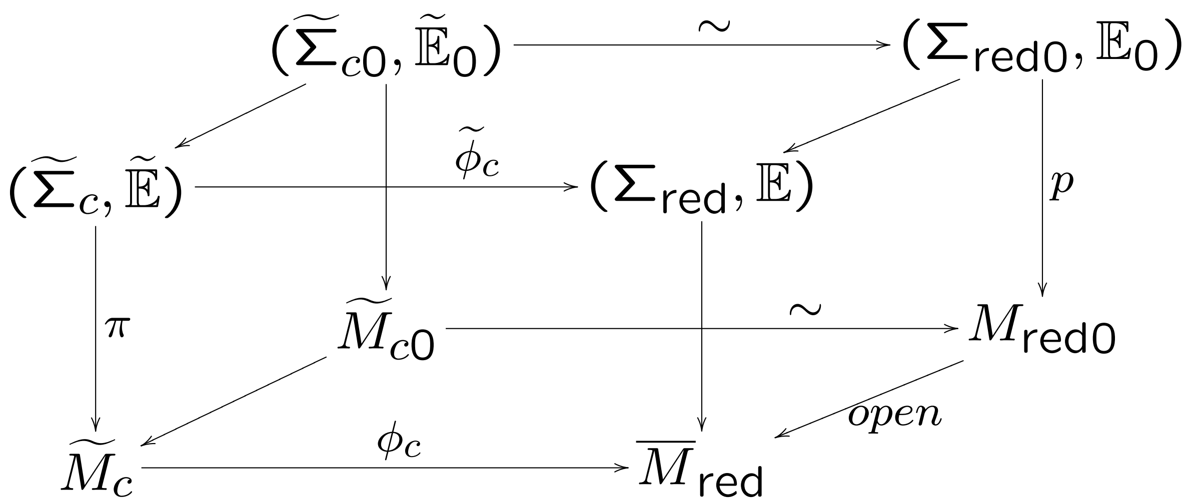
Standard resolution:

$$(T, p : T \times S \to T, \mathbb{L}, \mathbb{E}) \mapsto (\widetilde{T}, \pi : \widetilde{\mathbf{\Sigma}} \to \widetilde{T}, \widetilde{\mathbb{L}}, \widetilde{\mathbb{E}})$$

Let $\overline{M}$ carries a universal family of sheaves $\mathbb{E}$. The necessary condition for this is absence of strictly semistable sheaves with data $r, p(n)$.

Theorem 1. *There exist*
(1) projective variety $\widetilde{M}_c$,
(2) projective scheme $\widetilde{\Sigma}_c$ *together with flat morphism* $\widetilde{\Sigma}_c \xrightarrow{\pi} \widetilde{M}_c$, *whose fibres form a family of schemes over* $\widetilde{M}_c$,
(3) family of polarizations $\widetilde{\mathbb{L}}$ *on fibres of the family* $\widetilde{\Sigma_c}$, *s.t. Hilbert polynomial* $\chi(\widetilde{\mathbb{L}}^n|_{\pi^{-1}(\widetilde{y})})$ *of the fibre* $\pi^{-1}(\widetilde{y})$ *does not depend on the point* $\widetilde{y} \in \widetilde{M}_c$,
(4) locally free sheaf $\widetilde{\mathbb{E}}$ *on the scheme* $\widetilde{\Sigma}_c$,
(5) morphism $\phi_c : \widetilde{M}_c \to \overline{M}_{\mathrm{red}}$,
(6) morphism of families $\widetilde{\phi}_c : \widetilde{\Sigma}_c \to \Sigma_{\mathrm{red}}$,
s.t.
i) the morphism ϕ_c *is birational,*
ii) the variety $\widetilde{M}_c$ *contains open subset* $\widetilde{M}_{c0}$, *s.t. the restriction* $\phi_c|_{\widetilde{M}_{c0}} : \widetilde{M}_{c0} \to M_{\mathrm{red}0}$ *is an isomorphism,*
iii) the morphism $\widetilde{\phi}_c$ *is birational,*
iv) the morphism $\widetilde{\phi}_c$ *maps open subset* $\widetilde{\Sigma}_{c0} = \pi^{-1}\widetilde{M}_{c0}$ *isomorphically onto the subset* $\Sigma_{\mathrm{red}0}$,
v) there is a sheaf equality $(\widetilde{\phi}_{c*}\widetilde{\mathbb{E}})^{\vee\vee} = \mathbb{E}$.

This means that there is a commutative diagram of flat families



where all slanted arrows are open immersions and all edges except rectangles are fibred.

Theorem 2. *(i) There exists a sheaf of ideals* $\mathbb{J} \subset \mathcal{O}_{\widetilde{M}_c \times S}$ *s.t. the projection* $\pi : \widetilde{\Sigma}_c \to \widetilde{M}_c$ *can be expressed as a composite*

$$\pi : \widetilde{\Sigma}_c \xrightarrow{\Phi} \widetilde{M}_c \times S \xrightarrow{p} \widetilde{M}_c,$$

where Φ *is a morphism of blowing up of the sheaf of ideals* $\mathbb{J}$ *and* p *is a projection on the direct factor.*
(ii) The fibre of the projection π *over general point* $\widetilde{y} \in \widetilde{M}_{c0}$ *is isomorphic to the surface* S*. The fibre over special point* $\widetilde{y} \in \widetilde{M}_c \setminus \widetilde{M}_{c0}$ *is a reducible scheme. It contains component isomorphic to the blowing up the surface* S *in the sheaf of zeroth Fitting ideals* $\mathcal{F}itt^0(\mathcal{E}xt^1(E_{\phi_c(\widetilde{y})}, \mathcal{O}_S))$*.*

Hence there is a commutative diagram

$$\begin{array}{ccccc}
(\widetilde{\Sigma}_c, \widetilde{\mathbb{E}}) & \xrightarrow{\widehat{\pi}} & (\widehat{\Sigma}, \widehat{\mathbb{E}}) & \xrightarrow{\sigma} & (\Sigma_{\mathrm{red}}, \mathbb{E}) \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle p} \\
\widetilde{M}_c & \xrightarrow{\phi_c} & \overline{M}_{\mathrm{red}} & \xrightarrow{=} & \overline{M}_{\mathrm{red}}
\end{array}$$

Remark. The construction described involves the choice of an ample invertible sheaf $\widehat{\mathcal{L}}^{\otimes m}$ on intermediate blowup $\widehat{\Sigma}$. Although it is proven that for $m \gg 0$ the compactification $\widetilde{M}_c$ does not depend on the choice of $\widehat{\mathcal{L}}$.

The analogous construction was performed in the case when there exist strictly semistable torsion-free sheaves. This means that the Gieseker – Maruyama scheme cannot carry universal family of sheaves. In such a situation one has to work with pseudofamily. Its base is an étale covering of an appropriate birational preimage of Gieseker – Maruyama moduli variety. We elaborated a version of standard resolution for families with quasi-projective (and not projective) base. The desired compactification $\widetilde{M}_c$ arises as an algebraic space and we prove that it is a projective algebraic scheme. Constructions and results of this part are done over $\mathbb{C}$. The reason is technical: it was necessary to use the results of F. C. Kirwan obtained over $\mathbb{C}$ and (as it is known to me) having no analogues for arbitraty algebraically closed fields of zero characteristic.

Proposition. Pseudofamily of coherent sheaves exists for an appropriate birational preimage of any coarse moduli space of stable sheaves.

Definition[Ellingsrud G., Göttsche L., 1995] A *pseudofamily of sheaves* on the surface S, parameterized by the scheme X, consists of the following data:
(i) étale covering $\{B_i, \beta_i : B_i \to X\}_{i\in I}$,
(ii) collection $\{\mathbb{E}_i\}_{i\in I}$ of coherent $\mathcal{O}_{B_i\times S}$-sheaves, flat over B_i,
s.t.
for any two elements of the covering B_i, $B_{i'}$ there is $\mathcal{O}_{B_i\times_X B_{i'}}$-linear bundle $L_{ii'}$, s.t. for fibred product

$$\begin{array}{ccc} B_i \times_X B_{i'} & \xrightarrow{\beta^{i'}} & B_i \\ \beta^i \downarrow & & \downarrow \beta_i \\ B_{i'} & \xrightarrow{\beta_{i'}} & X \end{array}$$

there is an isomorphism of sheaves $(\beta^i, \mathrm{id}_S)^*\mathbb{E}_{i'} = (\beta^{i'}, \mathrm{id}_S)^*\mathbb{E}_i \otimes p^*L_{ii'}$.

Let $\mathbb{E} = \{B_i,\ \beta_i : B_i \to \overline{M}_{\mathsf{red}},\ \mathbb{E}_i\}_{i\in I}$ be a pseudofamily of sheaves on the scheme $\overline{M}_{\mathsf{red}} \times S$ and $E_{i,y}$ be a sheaf corresponding to the point $y \in B_i$, i.e. $E_{i,y} = \mathbb{E}_i|_{\{y\}\times S}$. Denote by $\mathbb{E}_0 = \{B_{0i},\ \beta_{0i} : B_{0i} \to M_{\mathsf{red}0},\ \mathbb{E}_{0i}\}_{i\in I}$ the pseudofamily of locally free sheaves which is defined as follows:

$$\begin{aligned} B_{0i} &:= \beta_i^{-1}(\beta_i(B_i) \cap M_{\mathsf{red}0}) \subset B_i, \\ \beta_{0i} &:= \beta_i|_{B_{0i}}, \\ \mathbb{E}_{0i} &:= \mathbb{E}_i|_{B_{0i}\times S}. \end{aligned}$$

Denote $\Sigma_{\mathsf{red},i} := B_i \times S$ and $\Sigma_{0i} := B_{0i} \times S$. Let $p_i : \Sigma_{\mathsf{red},i} \to B_i$ be the natural projection.

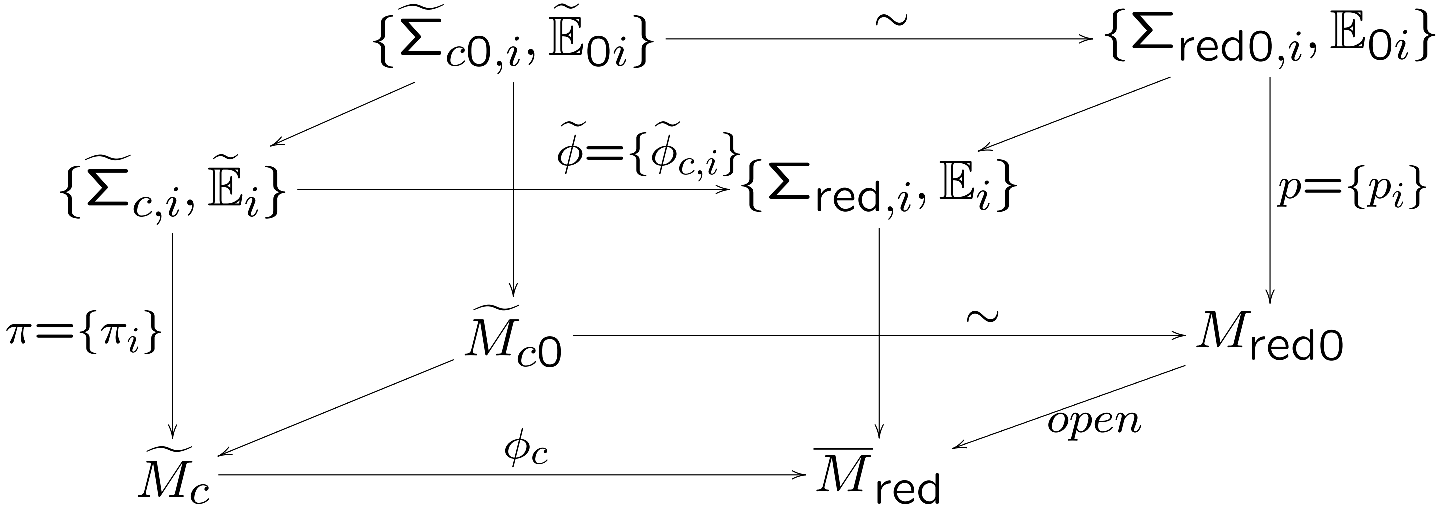
Theorem 3. *There exist*
(1) $\widetilde{M}_c$ *a projective algebraic variety,*
(2) étale covering $\{\widetilde{B}_i, \widetilde{\beta}_i : \widetilde{B}_i \to \widetilde{M}_c\}_{i\in I}$ *by quasi-projective varieties,*
(3) $\{\widetilde{\Sigma}_{c,i}\}_{i\in I}$ *a collection of quasi-projective schemes,*
(4) a collection of morphisms $\{\pi_i : \widetilde{\Sigma}_{c,i} \to \widetilde{B}_i\}_{i\in I}$*, which are flat of relative dimension 2 over their images,*
(5) a collection of families $\{\widetilde{\mathbb{L}}_i\}_{i\in I}$ *of polarizations on fibres of each* $\widetilde{\Sigma}_{c,i}$*, s.t. for every* i *the Hilbert polynomial* $\chi(\widetilde{\mathbb{L}}_i^n|_{\pi_i^{-1}(\widetilde{y})})$ *of the fibre* $\pi_i^{-1}(\widetilde{y})$ *does not depend on the point* $\widetilde{y} \in \widetilde{B}_i$*,*
(6) a collection of locally free sheaves $\{\widetilde{\mathbb{E}}_i\}_{i\in I}$ *on schemes* $\widetilde{\Sigma}_{c,i}$*,*
(7) a morphism of algebraic schemes $\phi_c : \widetilde{M}_c \to \overline{M}_{\mathrm{red}}$*,*
(8) morphisms of covering schemes $\phi_i : \widetilde{B}_i \to B_i$*,*
(9) morphisms of families $\widetilde{\phi}_{c,i} : \widetilde{\Sigma}_{c,i} \to \Sigma_{\mathrm{red},i}$*,*
s.t.

i) the morphism ϕ_c is birational and projective,
ii) the scheme $\widetilde{M}_c$ contains an open subscheme $\widetilde{M}_{c0}$, s.t. restriction $\phi|_{\widetilde{M}_{c0}}$: $\widetilde{M}_{c0} \to M_{\mathrm{red}0}$ is an isomorphism,
iii) morphisms ϕ_i are birational and projective,
iv) each scheme $\widetilde{B}_i$ contains an open subscheme $\widetilde{B}_{0i}$, s.t. the restriction $\phi_i|_{\widetilde{B}_{0i}}$ is an isomorphism,
v) the diagram

$$\begin{array}{ccc} \widetilde{B}_i & \xrightarrow{\phi_i} & B_i \\ {\scriptstyle \widetilde{\beta}_i}\downarrow & & \downarrow{\scriptstyle \beta_i} \\ \widetilde{M}_c & \xrightarrow{\phi_c} & \overline{M}_{\mathrm{red}} \end{array}$$

commutes,
vi) morphisms $\widetilde{\phi}_{c,i}$ are birational,
vii) each morphism $\widetilde{\phi}_{c,i}$ maps open subset $\widetilde{\Sigma}_{c0,i} = \pi_i^{-1}\widetilde{B}_{0i}$ isomorphically onto the subscheme $\Sigma_{\mathrm{red}0,i}$,
viii) there is an isomorphism of pseudofamilies of sheaves given by the formula $(\widetilde{\phi}_{i*}\widetilde{\mathbb{E}}_i)^{\vee\vee} = \mathbb{E}_i$.

This means that we have a commutative diagram of flat families of schemes equipped with pseudofamilies of sheaves



where all slanted arrows are open immersions and all edges except rectangles are fibred for each i.
There is a series of commutative diagrams:

$$\begin{array}{ccccc}
\{\widetilde{\Sigma}_{c,i}, \widetilde{\mathbb{E}}_i\} & \xrightarrow{\{\widehat{\pi}_i\}} & \{\widehat{\Sigma}_i, \widehat{\mathbb{E}}_i\} & \xrightarrow{\{\sigma_i\}} & \{\Sigma_{\mathrm{red},i}, \mathbb{E}_i\} \\
\downarrow \pi & & \downarrow & & \downarrow p \\
\widetilde{M}_c & \xrightarrow{\phi_c} & \overline{M}_{\mathrm{red}} & = & \overline{M}_{\mathrm{red}}
\end{array}$$

Pairs $((\widetilde{S},\widetilde{L}),\widetilde{E})$ consisting of $\widetilde{S},\widetilde{L},\widetilde{E}$ arising in standard resolution, are called *dS-pairs*. If $\widetilde{S}=S$, then such a pair $((\widetilde{S},\widetilde{L}),\widetilde{E})$ is called *S-pair*.

Length $l(\varkappa)$ of Artinian sheaf $\varkappa$ is defined as $l(\varkappa)=\chi(\varkappa)$. For zero-dimensional subscheme $Z\subset S$ one has $l(Z)=l(\mathcal{O}_Z)=\chi(\mathcal{O}_Z)$. Grothendieck's Quot-scheme of zero-dimensional quotient sheaves of length l of $\mathcal{O}_S$-sheaf F on the surface S is denoted as $\mathrm{Quot}^l F$. Quotient sheaf $q: F\twoheadrightarrow \varkappa$ corresponds to a point $q\in \mathrm{Quot}^l F$.

Theorem 4. *The fibre of the family $\pi:\widetilde{\Sigma}_c\to \widetilde{M}_c$ at a point $\widetilde{y}\in \widetilde{M}_c$*
i) is isomorphic to S if $\widetilde{y}\in \widetilde{M}'_{c0}$,
ii) is contained in the class of all $\mathrm{Proj}\bigoplus_{s\ge 0}(I[t]+(t))^s/(t^{s+1})$ *for $I=\mathcal{F}itt^0\mathcal{E}xt^2(\varkappa,\mathcal{O}_S)$, where $\varkappa$ is Artinian sheaf of length l which is a quotient sheaf of the direct sum $\mathcal{O}_S^{\oplus r}$, $l\le c_2$, if $\widetilde{y}\in \widetilde{M}_c\setminus\widetilde{M}_{c0}$.*

Definition. *S-stable (S-semistable) pair* $((\widetilde S,\widetilde L),\widetilde E)$ is the following data:
• $\widetilde S=\bigcup_{i\ge 0}\widetilde S_i$ – admissible scheme, $\sigma:\widetilde S\to S$ – canonical morphism, $\sigma_i:\widetilde S_i\to S$ – its restrictions on components $\widetilde S_i$, $i\ge 0$;
• $\widetilde E$ – vector bundle on the scheme $\widetilde S$;
• $\widetilde L\in \operatorname{Pic}\widetilde S$ – distinguished polarization of the form $\widetilde L=L\otimes(\sigma^{-1}I\cdot\mathcal O_{\widetilde S})$;
s.t.
• $\chi(\widetilde E\otimes\widetilde L^m)=rp(m)$;
• on the scheme $\widetilde S$ the sheaf $\widetilde E$ is *Gieseker-stable (Gieseker-semistable)* , i.e. for any proper subsheaf $\widetilde F\subset\widetilde E$ for $m\gg 0$

$$\frac{h^0(\widetilde F\otimes\widetilde L^m)}{\operatorname{rank}F}<\frac{h^0(\widetilde E\otimes\widetilde L^m)}{\operatorname{rank}E},$$

$$(\text{resp., }\frac{h^0(\widetilde F\otimes\widetilde L^m)}{\operatorname{rank}F}\le\frac{h^0(\widetilde E\otimes\widetilde L^m)}{\operatorname{rank}E});$$

• on each of additional components $\widetilde S_i, i>0$, the sheaf $\widetilde E_i:=\widetilde E|_{\widetilde S_i}$ is *quasi-ideal*, i.e. has a description $\widetilde E_i=\sigma_i^*\ker q_0/(\text{tors}\,|_{\widetilde S_i})$ for some $q_0\in\bigsqcup_{l\le c_2}\operatorname{Quot}^l\bigoplus^r\mathcal O_S$.

Subsheaf tors plays the role which is analogous to the role of torsion subsheaf on reduced scheme.
If $\widetilde{S} \cong S$, then S-stability (S-semistability) of the pair $(\widetilde{S}, \widetilde{E})$ is equivalent to Gieseker-stability (Gieseker-semistability) of vector bundle $\widetilde{E}$ on the surface $\widetilde{S}$ w.r.t. the polarization $\widetilde{L} \in \operatorname{Pic} \widetilde{S}$.

There is an isomorphism

$$\upsilon : H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m) \xrightarrow{\sim} H^0(S, E \otimes L^m)$$

of spaces of global sections. It is induced by the construction of resolution and is of use in the proof of the relation of semistability notions.

Theorem 5. *Let locally free $\mathcal{O}_S$-sheaf $\widetilde{E}$ arises by standard resolution from the coherent $\mathcal{O}_S$-sheaf E. The sheaf $\widetilde{E}$ is stable (semistable) on $\widetilde{S}$ w.r.t. distinguished polarization $\widetilde{L}$ iff the sheaf E is Gieseker-stable (Gieseker-semistable) w.r.t. the polarization L.*

The family of schemes $\pi : \widetilde{\Sigma} \to T$ is **birationally S-trivial** if there exist isomorphic open subschemes $\widetilde{\Sigma}_0 \subseteq \widetilde{\Sigma}$ and $\Sigma_0 \subseteq \Sigma = T \times S$, and there is a scheme equality $\pi(\widetilde{\Sigma}_0) = T$.

Let T be a scheme over the field k. Consider families of semistable pairs

$$\mathfrak{F}_T = \left\{ \begin{array}{l} \pi : \widetilde{\Sigma} \to T \text{ birationally} S-\text{trivial}, \\ \widetilde{\mathbb{L}} \in \text{Pic}\, \widetilde{\Sigma} \text{ flat over } \quad T, \\ \widetilde{\mathbb{L}} \text{ ample relatively to } T, \\ (\pi^{-1}(t), \widetilde{L}_t) \text{ admissible scheme} \\ \text{with distinguished polarization;} \\ \chi(\widetilde{L}_t^n) \text{ does not depend on } t, \\ \widetilde{\mathbb{E}} - \text{locally free } \mathcal{O}_{\widetilde{\Sigma}} - \text{sheaf, flat over } T; \\ \chi(\widetilde{\mathbb{E}} \otimes \widetilde{\mathbb{L}}^n)|_{\pi^{-1}(t)}) = rp(n); \\ ((\pi^{-1}(t), \widetilde{L}_t), \widetilde{\mathbb{E}}|_{\pi^{-1}(t)}) - \text{stable (semistable) pair} \end{array} \right\}$$

and a functor $\mathfrak{f} : (Schemes_k)^o \to (Sets)$ from the category of k-schemes to the category of sets which takes a scheme T to the set of equivalence classes $(\mathfrak{F}_T/\sim)$.

The equivalence relation $\sim$ is defined as follows.
Families $((\pi : \widetilde{\Sigma} \to T, \widetilde{\mathbb{L}}), \widetilde{\mathbb{E}})$ и $((\pi' : \widetilde{\Sigma}' \to T, \widetilde{\mathbb{L}}'), \widetilde{\mathbb{E}}')$ from the class $\mathfrak{F}$ are equivalent (notation: $((\pi : \widetilde{\Sigma} \to T, \widetilde{\mathbb{L}}), \widetilde{\mathbb{E}}) \sim ((\pi' : \widetilde{\Sigma}' \to T, \widetilde{\mathbb{L}}'), \widetilde{\mathbb{E}}'))$ if
1) there exist an isomorphism $\iota : \widetilde{\Sigma} \xrightarrow{\sim} \widetilde{\Sigma}'$ s.t. the diagram

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow[\sim]{\iota} & \widetilde{\Sigma}' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & T & \end{array}$$

commutes.
2) There exist linebundles L', L'' on T s.t. $\iota^* \widetilde{\mathbb{E}}' = \widetilde{\mathbb{E}} \otimes \pi^* L'$, $\iota^* \widetilde{\mathbb{L}}' = \widetilde{\mathbb{L}} \otimes \pi^* L''$.

The scheme $\widetilde{M}$ is a *coarse moduli space* for the functor $\mathfrak{f}$ if $\mathfrak{f}$ is corepresented by the scheme $\widetilde{M}$.

Since first the construction of reduced compactification $\widetilde{M}_{\mathsf{red}}$ is done, in $(Schemes_k)$ (and $(Schemes_k)^o$) the full subcategory $(RSch_k)$ (resp., $(RSch_k)^o$), of reduced schemes, is taken. Also we consider the restriction of the functor $\mathfrak{f}$ on subcategory $(RSch_k)^o$. This restriction is denoted $\mathfrak{f}_{\mathsf{red}}$.

Theorem 6. *The functor $\mathfrak{f}_{\mathrm{red}}$ has a coarse moduli space $\widetilde{M}_{\mathrm{red}}$ with following properties:*

(i) $\widetilde{M}_{\mathrm{red}}$ – projective Noetherian algebraic scheme with reduced structure;

(ii) there is a birational morphism of the union of main components of Gieseker – Maruyama scheme for the surface S and Hilbert polynomial $rp(m)$: $\kappa : \overline{M}_{\mathrm{red}} \to \widetilde{M}_{\mathrm{red}}$;

(iii) there is a birational morphism of constructive compactification: $\phi_r : \widetilde{M}_c \to \widetilde{M}_{\mathrm{red}}$;

(iv) there is a commutative triangle of compactifications:

$$\begin{array}{ccccc} & & \widetilde{M}_c & & \\ & \overset{\phi_c}{\swarrow} & & \overset{\phi_r}{\searrow} & \\ \overline{M}_{\mathrm{red}} & & \xrightarrow{\quad\kappa\quad} & & \widetilde{M}_{\mathrm{red}} \end{array} \tag{1}$$

(v) there is Zariski-open subscheme $\widetilde{M}_{\mathrm{red}0} \subset \widetilde{M}_{\mathrm{red}}$, corresponding to such pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ that $(\widetilde{S}, \widetilde{L}) \cong (S, L)$. Over $\widetilde{M}_{\mathrm{red}0}$ morphisms of the diagram (1) are isomorphisms, i.e. $M_{\mathrm{red}0} \cong \widetilde{M}_{c0} \cong \widetilde{M}_{\mathrm{red}0}$;

(vi) there is M-equivalence relation defined on the class of semistable pairs, s.t. pairs are represented by the same point in $\widetilde{M}_{\mathrm{red}}$ iff they are M-equivalent.

All reasonings are applicable to any Hilbert polynomials with no relation to the value of discriminant as well as to the number and geometry of irreducible components of Gieseker – Maruyama scheme. In general (reducible) case the theorem provides existence of a coarse moduli space for any maximal under inclusion irreducible substack in $\coprod(\mathfrak{F}_T/\sim)$, which contains such pairs $((\pi^{-1}(t), \widetilde{L}_t), \widetilde{\mathbb{E}}|_{\pi^{-1}(t)})$ that $(\pi^{-1}(t), \widetilde{L}_t) \cong (S, L)$. These pairs were called S-pairs. We mean under $\widetilde{M}_{\mathrm{red}}$ namely the moduli space of the irreducible substack which contains S-pairs.

Theorem 7. *The functor* $\mathfrak{f}$ *has a coarse moduli space* $\widetilde{M}$ *which is a projective Noetherian algebraic scheme of finite type. The scheme* $\widetilde{M}$ *contains open subscheme* $\widetilde{M}_0$ *which is isomorphic to the open subscheme* M_0' *in the Gieseker – Maruyama scheme* $\overline{M}$ *corresponding to the same data* $r, p(n)$.

The theorem guarantees the existence of a coarse moduli space for any maximal irreducible substack in $\coprod(\mathfrak{F}_T/\sim)$, which contains S-pairs. The mentioned substack contains families (with possibly nonreduced base schemes) consisting of those and only those semistable pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ that satisfy the condition:

there exist a family of semistable pairs $((\widetilde{\Sigma}_T, \widetilde{\mathbb{L}}_T), \widetilde{\mathbb{E}}_T)$ with reduced irreducible base T, containing the pair $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ and at least one S-pair.

We mention under $\widetilde{M}$ the moduli space of the described substack.

Definition. The flat family of stable pairs $((\widetilde{\Sigma}^s, \widetilde{\mathbb{L}}^s), \widetilde{\mathbb{E}}^s)$, with a projection $\pi' : \widetilde{\Sigma}^s \to \widetilde{M}^s$, is called *universal*, if:
for a flat family $((\widetilde{p} : \widetilde{\Sigma}_T \to T, \mathbb{L}_T), \mathbb{F}_T)$ of stable pairs with base T, s.t. for $n \gg 0$ $\chi(\widetilde{\mathbb{L}}^n_T|_{\widetilde{p}^{-1}(t)})$ does not depend on $t \in T$, $\chi(\mathbb{F}_T \otimes \widetilde{\mathbb{L}}^n_T|_{\widetilde{p}^{-1}(t)}) = rp(n)$ and $\widetilde{\mathbb{L}}_T|_{\widetilde{p}^{-1}(t)}$ is the distinguished polarization on the fibre $\widetilde{p}^{-1}(t)$

• there are induced morphisms $\mu_{\mathbb{F}} : T \to \widetilde{M}^s$ и $\widetilde{\mu}_{\mathbb{F}}$ s.t. the square

$$\begin{array}{ccc} \widetilde{\Sigma}_T & \xrightarrow{\widetilde{\mu}_{\mathbb{F}}} & \widetilde{\Sigma}^s \\ {\scriptstyle \widetilde{p}}\downarrow & & \downarrow{\scriptstyle \pi'} \\ T & \xrightarrow{\mu_{\mathbb{F}}} & \widetilde{M}^s \end{array}$$

is Cartesian;

• there exist linear bundles L', L'' on the scheme T s.t. $\mathbb{F}_T \otimes \widetilde{p}^* L' = \widetilde{\mu}^*_{\mathbb{F}} \widetilde{\mathbb{E}}^s$ и $\widetilde{\mathbb{L}}_T \otimes \widetilde{p}^* L'' = \widetilde{\mu}^*_{\mathbb{F}} \widetilde{\mathbb{L}}^s$.

Definition. *Pseudofamily of admissible semistable pairs* consists of

1) schemes B_i,
2) étale morphisms $\beta_i : B_i \to \widetilde{M}$,
3) schemes $\widetilde{\Sigma}_i$,
4) flat morphisms of schemes $\pi_i : \widetilde{\Sigma}_i \to B_i$,
5) ample invertible sheaves $\widetilde{\mathbb{L}}_i$ of $\mathcal{O}_{\widetilde{\Sigma}_i}$-modules,
6) locally free sheaves $\widetilde{\mathbb{E}}_i$ of $\mathcal{O}_{\widetilde{\Sigma}_i}$-modules

s.t.

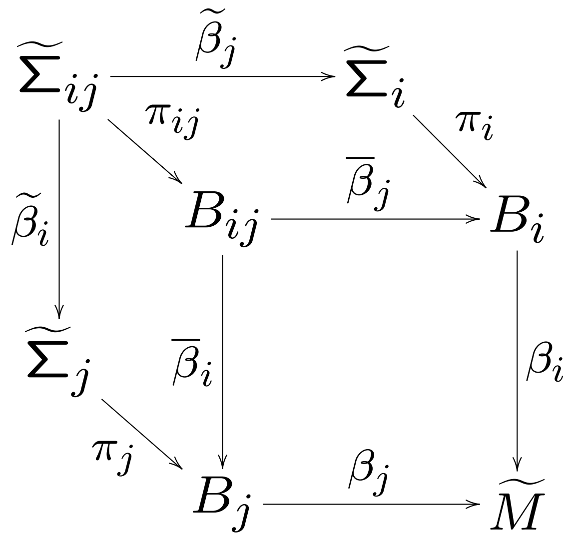
- morphisms β_i form an étale covering of the scheme $\widetilde{M}$,
- for each closed point $b \in B_i$ a colection $((\pi_i^{-1}(b), \widetilde{\mathbb{L}}_i|_{\pi_i^{-1}(b)}), \widetilde{\mathbb{E}}_i|_{\pi_i^{-1}(b)})$ is a semistable admissible pair,

and the following *glueing conditions* hold: for any pair of indices $i \neq j$ set $B_{ij} := B_i \times_{\widetilde{M}} B_j$; let $B_i \stackrel{\overline{\beta}_j}{\leftarrow} B_{ij} \stackrel{\overline{\beta}_i}{\to} B_j$ be projections of fibred product.

Then there exist

• scheme isomorphisms $B_{ij} \times_{B_j} \widetilde{\widetilde{\Sigma}}_j \cong \widetilde{\widetilde{\Sigma}}_i \times_{B_i} B_{ij}$.

Other notations are fixed by the fibred diagram



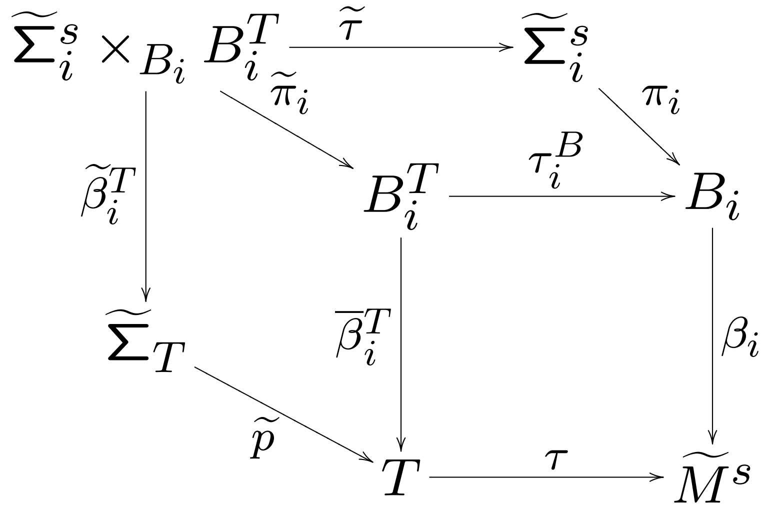
• invertible $\mathcal{O}_{B_{ij}}$-sheaves L'_{ij} и L''_{ij} s.t.

$$\widetilde{\beta}_i^* \widetilde{\mathbb{E}}_i \cong \widetilde{\beta}_j^* \widetilde{\mathbb{E}}_j \otimes \pi_{ij}^* L'_{ij}; \quad \widetilde{\beta}_i^* \widetilde{\mathbb{L}}_i \cong \widetilde{\beta}_j^* \widetilde{\mathbb{L}}_j \otimes \pi_{ij}^* L''_{ij}.$$

If we are given two pseudofamilies then referring index i in the glueing conditions to the first pseudofamily and j to the second one, we arrive to the definition of *equivalent pseudofamilies*.

Notation: $B_i^T := B_i \times_{\widetilde{M}^s} T$, $\tau_i^B : B_i^T \to B_i$, $\overline{\beta}_i^T : B_i^T \to T$ are projections of fibred product.

Definition. The pseudofamily $(\beta_i : B_i \to \widetilde{M}^s, \pi_i : \widetilde{\Sigma}_i^s \to B_i, \widetilde{\mathbb{L}}_i^s, \widetilde{\mathbb{E}}_i^s)$ is *universal* if for any T-based family $(T, \widetilde{\Sigma}_T, \widetilde{\mathbb{L}}_T, \widetilde{\mathbb{E}}_T)$ of stable admissible pairs there are a morphism of schemes $\tau : T \to \widetilde{M}^s$ and isomorphisms $\widetilde{\Sigma}_i^s \times_{B_i} B_i^T \cong B_i^T \times_T \widetilde{\Sigma}_T$. In notation defined by the fibred diagram



for appropriate invertible sheaves of $\mathcal{O}_{B_i^T}$-modules ${L'}_i^T$ and ${L''}_i^T$ there are isomorphisms $\widetilde{\tau}^* \widetilde{\mathbb{E}}_i^s \cong \widetilde{\beta}_i^{T*} \widetilde{\mathbb{E}}_T \otimes \widetilde{\pi}_i^* {L'}_i^T$; $\quad \widetilde{\tau}^* \widetilde{\mathbb{L}}_i^s \cong \widetilde{\beta}_i^{T*} \widetilde{\mathbb{L}}_T \otimes \widetilde{\pi}_i^* {L''}_i^T$.

Theorem 8. *Let for all semistable admissible pairs there exists such $m \gg 0$ that induced immersions*

$$j : \widetilde{S} \hookrightarrow G(H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m), r)$$

have no nontrivial $PGL(H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m))$-automorphisms. The acheme $\widetilde{M}$ contains open subscheme $\widetilde{M}^s$ which carries a family $((\widetilde{\Sigma}^s, \widetilde{\mathbb{L}}^s), \widetilde{\mathbb{E}}^s)$ with universal property. This subscheme as a set corresponds to the image $\kappa(\overline{M}^s_{\mathrm{red}})$, where $\overline{M}^s_{\mathrm{red}}$ is open subscheme of $PGL(H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m))$-stable points in the reduced Gieseker – Maruyama scheme $\overline{M}_{\mathrm{red}}$. If the scheme $\overline{M}$ carries a universal family of stable coherent sheaves then the scheme $\widetilde{M}$ also has a universal family $((\widetilde{\Sigma}, \widetilde{\mathbb{L}}), \widetilde{\mathbb{E}})$.

Theorem 9. *The scheme* $\widetilde{M}$ *contains an open subscheme* $\widetilde{M}^s$ *which carries pseudofamily* $(((\beta_i^s : B_i^s \to \widetilde{M}^s, \widetilde{\Sigma}_i^s, \widetilde{\mathbb{L}}_i^s), \widetilde{\mathbb{E}}_i^s)$ *with universal property. This subscheme as a set corresponds to the image* $\kappa(\overline{M}^s_{\text{red}})$, *where* $\overline{M}^s_{\text{red}}$ *is open subscheme of* $PGL(H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m))$*-stable points in* $\overline{M}_{\text{red}}$. *If the scheme* $\overline{M}$ *carries a universal family of stable coherent sheaves then* $\widetilde{M}$ *has universal pseudofamily* $((\beta_i : B_i \to \widetilde{M}, \widetilde{\Sigma}_i, \widetilde{\mathbb{L}}_i), \widetilde{\mathbb{E}}_i)$.

In particular, for $\widetilde{M}$ there is an analog of the numerical condition for existence of the universal family proved by Maruyama. Hilbert polynomial $rp(n)$ can be rewritten in the form

$$rp(n) = \sum_{i=0}^{2} a_i \binom{n+i}{i},$$

where a_0, a_1, a_2 are integers. Let $\delta(a_0, a_1, a_2)$ be their greatest common divisor.

Corollary. *Let $\delta(a_0, a_1, a_2) = 1$. Then $\widetilde{M}$ carries universal pseudofamily. If for all admissible stable pairs there is $m \gg 0$ s.t. induced immersions*

$$j : \widetilde{S} \hookrightarrow G(H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m), r)$$

have no nontrivial $PGL(H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m))$-automorphisms, then $\widetilde{M}$ carries universal family.

The Gieseker – Maruyama functor $\mathfrak{f}^{GM} : (Schemes_k)^o \to Sets$ is defined as follows: $T \mapsto \{\mathfrak{F}_T^{GM}\}/\sim$, where

$$\mathfrak{F}_T^{GM} = \left\{ \begin{array}{l} \mathbb{E}_T\text{– sheaf of } \mathcal{O}_{T\times S}-\text{modules flat over } T; \\ \mathbb{L}_T\text{– invertible sheaf of } \mathcal{O}_{T\times S}-\text{modules,} \\ \text{very ample relative to } T; \\ E_t := \mathbb{E}_T|_{t\times S} \text{ torsion-free and Gieseker-semistable} \\ \ \text{w.r.t. } L_t := \mathbb{L}_T|_{t\times S}; \\ \chi(E_t \otimes L_t^m) = rp(m). \end{array} \right\}$$

Families $(\mathbb{L}_T, \mathbb{E}_T)$ и $(\mathbb{L}'_T, \mathbb{E}'_T)$ are said to be equivalent if there are invertible $\mathcal{O}_T$-sheaves $\mathcal{L}'$ and $\mathcal{L}''$ s.t. for the projection $p : T \times S \to T$ one has $\mathbb{E}'_T \cong \mathbb{E}_T \otimes p^*\mathcal{L}'$ and $\mathbb{L}'_T \cong \mathbb{L}_T \otimes p^*\mathcal{L}''$.

Theorem 10. *There is a morphism of reduced moduli functors* $\underline{\tau}_{\text{red}} : \mathfrak{f}^{GM}_{\text{red}} \to \mathfrak{f}_{\text{red}}$*, defined by the procedure of standard resolution.*

Theorem 11. *Main components of reduced scheme* $\widetilde{M}_{\text{red}}$ *are isomorphic to main components of reduced Gieseker – Maruyama scheme.*

Theorem 12. *The Gieseker – Maruyama functor* $\mathfrak{f}^{GM}$ *of semistable torsion-free coherent sheaves of rank* r *and reduced Hilbert polynomial* $p(n)$ *on the surface* (S, L)*, has a natural transformation* $\underline{\kappa}$ *to the functor of admissible semistable pairs of the form* $((\widetilde{S}, \widetilde{L}), \widetilde{E})$*, where locally free sheaf* $\widetilde{E}$ *on the scheme* $(\widetilde{S}, \widetilde{L})$ *has same rank and reduced Hilbert polynomial. In particular there exists a morphism of moduli schemes* $\kappa : \overline{M} \to \widetilde{M}$ *related to the natural transformation* $\underline{\kappa}$*.*

Remark. The morphism $\kappa_{\text{red}} : \overline{M}_{\text{red}} \to \widetilde{M}_{\text{red}}$ constructed earlier is a reduction of the morphism κ.

Theorem 13(generalisation of flatness criterion) *Let a projective morphism $f : X \to T$ of Noetherian schemes of finite type is included into the commutative diagram*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathbb{P}^N_T \\ & {\scriptstyle f}\searrow & \downarrow \\ & & T \end{array}$$

Coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is flat relatively f (i.e. flat as a sheaf of $\mathcal{O}_T$-modules) iff for an invertible $\mathcal{O}_X$-sheaf $\mathcal{L}$ which is very ample relatively to T and s.t. $\mathcal{L} = i^*\mathcal{O}(1)$, for any closed point $t \in T$ and its m-th infinitesimal neighborhood $t^{(m)}$ the function

$$\varpi_t^{(m)}(\mathcal{F}, n) = \frac{\chi(\mathcal{F} \otimes \mathcal{L}^n|_{f^{-1}(t^{(m)})})}{\chi(\mathcal{O}_{t^{(m)}})}$$

does not depend on the choice of $t \in T$ and $m \in \mathbb{N}$.

Remark. If T is reduced then it is enough to investigate the case $n = 0$ what corresponds to the classical criterion $\varpi_t^{(0)}(\mathcal{O}_X, m) = P_t(m)$.

Theorem 13 is of use to prove

Theorem 14. *There is a natural transformation*

$$\underline{\tau} : \mathfrak{f} \to \mathfrak{f}^{GM}$$

of each maximal closed irreducible subfunctor of the moduli functor of admissible semistable pairs, which contains S-pairs, to the corresponding maximal closed irreducible subfunctor of the Gieseker – Maruyama moduli functor which contains locally free sheaves with same rank and Hilbert polynomial. This natural transformation is inverse to the natural transformation $\underline{\kappa}$ induced by the procedure of standard resolution. Then both morphisms of nonreduced moduli

$$\underline{\kappa} : \mathfrak{f}^{GM} \to \mathfrak{f} \quad \text{и} \quad \underline{\tau} : \mathfrak{f} \to \mathfrak{f}^{GM}$$

are mutually inverse isomorphisms. The union of main components of nonreduced moduli scheme $\widetilde{M}$ for the functor $\mathfrak{f}$ is isomorphic to the union of main components of nonreduced Gieseker – Maruyama scheme $\overline{M}$ for sheaves with same rank and Hilbert polynomial.

Open questions and directions of study

1. What about existence and geometry of those components of the scheme $\widetilde{M}$ which do not contain S-pairs?

2. Is there analog for Kobayashi – Hitchin correspondence on admissible schemes $\widetilde{S}$? What are interpretations of notions of connection and anti-self-duality condition in this case?

3. In case of positive answer for the previous question, how to interpret strictly semistable S-pairs? For example, is there a procedure of stabilization?

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THANK YOU!

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