Products of groups which contain abelian subgroups of finite index

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Factorized groups

A group G is called **factorized**, if

$$G = AB = \{ab \mid a \in A,\ b \in B\}$$

is the product of two subgroups A and B of G.

General problem.

What can be said about the structure of the factorized group G if the structures of its subgroups A and B are known?

Itô's Theorem

Theorem (N.Itô 1955).

If the group $G = AB$ is the product of two abelian subgroups A and B, then G is metabelian.

Remark. This theorem is unique in the following sense.

1. The statement is very precise, it is the basis for almost all known results about products of two abelian subgroups.

2. The proof is by a surprisingly short commutator calculation.

3. It seems to be almost impossible to generalize this argument to more general situations, for instance for products of two nilpotent groups (even of class two).

Products of abelian-by-finite groups

<u>MAIN PROBLEM</u>.

Let the group $G = AB$ be the product of two abelian-by-finite subgroups A and B (i.e. A and B have abelian subgroups of finite index).

Is G always soluble-by-finite or perhaps even metabelian-by-finite?

Remark. This is Question 3 in

[AFG] B.A., S.Franciosi, F. de Giovanni, "Products of groups", Oxford University Press (1992)

Some known results

This seemingly simple question is very difficult to attack.

It has a positive answer for **linear goups (Ya.Sysak 1986)** and for **residually finite groups (J.Wilson 1990)**.

Theorem (N.S.Chernikov 1981).

If the group $G = AB$ is the product of two central-by-finite subgroups A and B, then G is soluble-by-finite.

It is unknown whether G must be metabelin-by-finite in this case.

The Theorem of Kegel-Wielandt

<u>Theorem</u> (H. Wielandt 1953, O. Kegel 1961).

If the finite group $G = AB$ is the product of two nilpotent subgroups A and B, then G is soluble.

<u>Theorem</u> (L. Kazarin 1979).

If the finite group $G = AB$ is the product of two subgroups A and B, each of which possesses nilpotent subgroups of index at most 2, then G is soluble.

Some general remarks

Let the group $G = AB$ be the product of two subgroups A and B, which have (abelian) subgroup A_0 resp. B_0 of finite index $n =\mid A : A_0 \mid$ and $m =\mid B : B_0 \mid$.

By Lemma 1.2.5 of [AFG] **the subgroup $< A_0, B_0 >$ has finite index at most nm**.

Clearly, if also we should have **have that $A_0 B_0 = B_0 A_0$ is a subgroup of G, then G has a metabelian subgroup (by Itô) of finite index**.

Thus, if additional permutability conditions are imposed, some factorization problems become much easier and sometimes trivial.

(see "Products of finite groups" by A.Ballester-Bolinches, R.Esteban-Romero, M.Asaad, de Gruyter 2010)

Factorgroups and subgroups

If N be a normal subgroup of the factorized group $G = AB$, then clearly

$$G/N = (AN/N)(BN/N)$$

is likwise factorized by two epimorphic images AN/N of A resp. BN/N of B.

But in general it is very difficult to find subgroups S of G that inherit the factorization as $S = (A \cap S)(B \cap S)$.

A slight generalization of Itô's theorem

Lemma. Let N be a normal subgroup of a group G. Suppose that G contains two abelian subgroups X and Y such that $N \subseteq XY$. Then NX is metabelian.

Proof. The subgroup $NX = NX \cap XY = X(NX \cap Y)$ is metabelian by Itô.

Specializing the problem

Special case of the Main problem.

Let the group $G = AB$ is the product of two subgroups A and B, where A contains an abelian subgroup A_0 and B contains an abelian subgroup B_0 such that the indices $|A : A_0|$ and $|B : B_0|$ are at most 2.
Is then G soluble and/or metabelian-by-finite?

Such "index 2"-problems were considered for finite groups in the 1950'es, among others by B.Huppert and W.R.Scott.
V. Monakhov showed in 1974 that a finite group $G = AB$ is soluble if A and B have cyclic subgroups of index at most 2.

Products of infinite cyclic groups

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Theorem (P.Cohn 1956).

If the subgroups A and B are infinite cyclic, then $G = AB$ is metacyclic-by-finite; i.e. G has a metacyclic (normal) subgroup of finite index.

Products of cyclic-by-finite groups

The following theorem generalizes the results of P. Cohn and V. Monakhov.

Theorem 1 (B.A., Ya.Sysak, Arch. Math. 90 (2008), 101-111).

If the group $G = AB$ is the product of two subgroups A and B, each of which has a cyclic subgroup of index at most 2, then G is metacyclic-by-finite.

Remarks on the proof of Theorem 1

Note that a non-abelian infinite group which has a cyclic group of index 2 must be the **the infinite dihedral group**. This ensures the existence of involutions in this case.

Therefore we may use the existence of "enough" involutions which can be used for computations.

An important idea in the proof is to show that the normalizer in G of **an infinite cyclic subgroup of one of the factors A or B has a non-trivial intersection with the other factor**.

Involutions and Dihedral groups

An element $x \neq 1$ in a group G is called an **involution**, if $x^2 = 1$, i.e. $x = x^{-1}$

Dihedral groups.

A group is called **dihedral** if it can be generated by two distinct involutions.

The structure of dihedral groups

Let the dihedral group G be generated by the two involutions x and y. Let $c = xy$ and $C =< c >$. Then we have

a) The cyclic subgroup C is normal and of index 2 in G, the group $G = C \rtimes < i >$ is the semidirect product of C and a subgroup $< i >$ of order 2,

b) If G is non-abelian, then C is characteristic in G.

c) Every element of $G \setminus C$ is an involution which inverts every element of C, i.e. if $g \in G \setminus C$, then $c^g = c^{-1}$ for $c \in C$,

d) The set $G \setminus C$ is a single conjugacy class if and only if the order of C is finite and odd; it is the union of two conjugacy classes otherwise.

Locally dihedral groups

The group G is <u>**locally dihedral**</u> if it has a local system of dihedral subgroups, i.e. every finite subset of G is contained in some dihedral subgroup of G.

Every **periodic locally dihedral group** is locally finite and every finite subgroup of such a group is contained in a finite dihedral subgroup.

<u>**Lemma.**</u> Every periodic locally dihedral group G has a locally cylic normal subgroup C of index 2, and every element of $G \setminus C$ is an involution that inverts every element of C;

$G = C \rtimes < i >$ is the semidirect product of C and a subgroup $< i >$ of order 2.

Products of periodic locally dihedral groups

Theorem 2 (B.A., A.Fransman, L.Kazarin, J. Alg. 350 (2012), 308-317).

Every group $G = AB$ which is the product of two periodic locally dihedral subgroups A and B is soluble.

The proof depends to a large extend on methods and results about finite products.

Products of finite dihedral groups

A first step is to consider more thoroughly products of finite dihedral groups.

Proposition. **Let $G = AB$ be a finite group, which is a product of subgroups A and B, where A is dihedral and B is either cyclic or a dihedral group. Then $G^{(7)} = 1$.**

Some useful lemmas

Lemma. Let the finite group $G = AB$ be the product of two subgroups A and B, then for every prime p there exists a Sylow-p-subgroup of G which is a product of a Sylow-p-subgroup of A and a Sylow-p-subgroup of B.

Lemma. Let the finite group $G = AB$ be the product of subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0 B_0 = B_0 A_0$, then $A_0^x B_0 = B_0 A_0^x$ for all $x \in G$. Assume in addition that A_0 and B_0 are π-groups for a set of primes π. If $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.

Lemma. Let the locally finite group $G = AB$ be the product of two subgroups A and B, and let A_0 and B_0 be finite normal subgroups of A and B, respectively. Then there exists a finite subgroup E of G such that
$A_0, B_0 \subseteq E \subseteq N_G(< A_0, B_0 >)$ and $E = (A \cap E)(B \cap E)$.

Generalized dihedral groups

Definition. A group G is **generalized dihedral** if it is **of dihedral type**,
i.e. G contains an abelian subgroup X of index 2 and an involution τ which inverts every element in X.

Clearly $A = X \rtimes < a >$ is the semi-direct product of an abelian subgroup X and an involution a, so that $x^a = x^{-1}$ for each $x \in X$.

Properties of generalized dihedral groups

Let A be generalized dihedral. Then the following holds

1) every subgroup of X is normal in A;
2) if A is non-abelian, then every non-abelian normal subgroup of A contains the derived subgroup A' of A;
3) $A' = X^2$ and so the commutator factor group A/A' is an elementary abelian 2-group;
4) the center of A coincides with the set of all involutions of X;
5) the coset aX coincides with the set of all non-central involutions of A;
6) two involutions a and b in A are conjugate if and only if $ab^{-1} \in X^2$.
7) if A is non-abelian, then X is characteristic in A.

Products of generalized dihedral groups

Theorem 3 (B.A., Ya.Sysak, J. Group Theory 16 (2013), 299-318)

(a) Let the group $G = AB$ be the product of two subgroups A and B, each of which is either abelian or generalized dihedral. Then G is soluble.

(b) If, in addition, one of the two subgroups, B say, is abelian, then the derived length of G does not exceed 5.

Corollary. Let the group $G = AB$ be the product of two subgroups A and B, each of which contains a torsion-free locally cyclic subgroup of index at most 2. Then G is soluble and metabelian-by-finite.

Remarks on the proof of Theorem 3

The proof of this theorem is elementary and almost only uses computations with involutions. Extensive use is made by the fact that every two involutions of a group generate a dihedral subgroup.

A main idea of the proof is to show that

the normalizer in G of a non-trivial normal subgroup of one of the factors A or B has a non-trivial intersection with the other factor.

If this is not the case we may find commuting involutions in A and B and produce a nontrivial abelian normal subgroup by other computations.

A Reduction

Consider a group $G = AB$ with subgroups A and B such that $A = X \rtimes \langle c \rangle$ and $B = Y \rtimes \langle d \rangle$ for abelian subgroups X and Y and involutions c and d with $x^c = x^{-1}$ for every $x \in X$, $y^d = y^{-1}$ for every $y \in Y$. **Let 1 be the only abelian normal subgroup of G.** Then clearly $X \cap Y = 1$, so that $|A \cap B| \leq 2$.

Suppose that

$$N_A(\langle y \rangle) = 1 = N_B(\langle x \rangle)$$

for every $x \in X^\#$ and every $y \in Y^\#$. Then $A \cap B = 1$.

We show that there exist involutions $cx \in A$ and $yd \in B$ such that

$$(cx)(yd) = (yd)(cx) .$$

Therefore we may assume that $cd = dc$.

Lemma. There exists some $x \in X^{\#}$ and $y \in Y^{\#}$ so that xy is an involution and

$$(cd)(xy^{-1}) = xd\,.$$

Similarly we have

$$(dc)(yx^{-1}) = yc\,.$$

$\square$

Therefore

$$xd = yc \quad \text{and so} \quad xc = yd \in A \cap B = 1\,.$$

This **contradiction** shows that there exists a non-trivial normal subgroup $\langle x \rangle$ of A or $\langle y \rangle$ of B such that

$$N_A(\langle y \rangle) \neq 1 \quad \text{or} \quad N_B(\langle x \rangle) \neq 1\,.$$

An application

A group G is **saturated (or covered) by subgroups in a set** $\mathfrak{S}$ if every finite subgroup S of G is contained in subgroup of G which is isomorphic to a subgroup in $\mathfrak{S}$.

A.Shlopkin and A.Rubashkin proved in Algebra i Logika (2005) that **a locally finite group which is saturated by finite dihedral groups is locally dihedral**, and this also holds some classes of periodic groups.

Question. Is every periodic group saturated by dihedral subgroups locally dihedral?

A locally finite group which is saturated by dihedral subgroups is locally dihedral:

If x and y are two elements of G with $o(x) > 2$ and $o(y) > 2$, then the finite group $< x, y >$ is contained in a proper finite dihedral group $D =< a > \rtimes < i >$ (by saturation). Since $x \in < a >$, $y \in < a >$, it follows that $xy = yx$.

Therefore the elements of G with order more than 2 generate a locally cyclic normal subgroup H of G. Clearly the set $G \setminus H$ is non-empty and consists only of involutions.

Let $t \in G \setminus H$ a fixed and $x \in G \setminus H$ **an arbitrary involution.**

If $h \in H$ with $o(x) > 2$, then the finite subgroup $< h, x, t >$ is contained in a dihedral subgroup $D =< h_1 > \rtimes < t >$.

Then $h_1 \in H$ by the definition of H. Thus $x \in D \subseteq H \rtimes < t >$ for every involution $x \in G$. It follows that $G = H \rtimes < t >$.

Extension to arbitrary periodic groups

Assume there exists a periodic group G saturated by dihedral subgroups which is not locally dihedral.

By results in the paper by A.Shlopkin and A.Rubashkin G is not locally finite and the centralizer $C_G(\gamma)$ of every involution γ in G is a (finite or infinite) periodic locally dihedral group, and there exist at least two involutions τ and $\mu \neq \tau$ in a Sylow-2-subgroup S of G.

We show that $G = AB$ **where** $A = C_G(\tau)$ **and** $B = C_G(\mu)$ **are locally dihedral**.

By Theorem 2 or 3 G is soluble and so locally finite.
<u>CONTRADICTION!</u>

Saturated periodic groups

The contradiction proves the following

Theorem 4 (B.A., L.Kazarin, Proceedings Ischia Group Theory Conference 2010).

If the infinite periodic group G is saturated by finite dihedral subgroups, then G is a locally finite dihedral group

A conjecture

I. Lysenok has proved that there exists a group G such that every finite subgroup of G is contained in a direct product of finite dihedral subgroups.

Conjecture. **Let G be a group such that every finite subgroup of G is contained in a direct product of a fixed number of finite dihedral subgroups. Then G is soluble.**

Chernikov groups

Definition. An abelian-by-finite group with minimum condition on its subgroup is called a <u>**Chernikov group**</u>.

The <u>**finite residual**</u> $J = J(G)$ of a group G is the intersection of all subgroups of G with finite index

$$J(G) = \bigcap G/N, N \subseteq G, |G : N| < \infty$$

A group G is a Chernikov group if and only if

1. $J(G)$ is the direct product of finitely many quasicyclic (Prüfer) p-groups for finitely many primes p,
2. $G/J(G)$ is finite.

Induction parameters for Chernikov groups.

For a Chernikov group X define the parameter $\Theta(X) = (r, m)$ where

1. $r = r(X)$ is the number of quasicyclic (Prüfer) subgroups in a decomposition of the radicable abelian group $J(X)$ (the **rank** of $J(X)$)
2. $m = m(X) = |X : J(X)|$.

A linear ordering on the set of pairs (r, s) is given by $(r, s) < (r_1, s_1)$ if $r < r_1$ or $r = r_1$ and $s < s_1$.

If U is a subgroup of X, then $\Theta(U) \leq \Theta(X)$.
If $\Theta(U) = \Theta(X)$, then $U = X$.

(Generalized) soluble groups

N.F. Sesekin has shown in 1968 that every group $G = AB$ which is the product of two abelian subgroups A and B with minimum condition, also satisfies the minimum condition on all its subgroups and is therefore a metabelian Chernikov group.

<u>Theorem</u> (B.A., O.Kegel, N.S.Chernikov, ≈1972).

Let the group $G = AB$ be the product of two Chernikov subgroups A and B. If G is soluble or generalized soluble in some sense, then G is also a Chernikov group and we have

$$J(G) = J(A)J(B).$$

Products of Chernikov subgroups "with index at most 2"

Theorem 5 **(B.A., L.Kazarin, Israel J. Math. 175 (2010), 363-389).**
Let the group $G = AB$ be the product of two Chernikov subgroups A and B, which both have abelian subgroups A_0 and B_0 respectively with index at most 2.

Let further one of the two subgroups, A say, be of dihedral type, i.e. A contains an involution τ which inverts every element of A_0.

Then G is a soluble Chernikov group and $J(G) = J(A)J(B)$.

If the index of $J(A)$ in A is m and the index of $J(B)$ in B is n, then the index of $J(G)$ in G is not more than mn.

Trifactorized groups

Definition.

A group G is called **trifactorized** if

$$G = AB = AC = BC$$

for three subgroups A, B and C.

Remark.

Many proofs about factorized groups finally reduce to the consideration of groups of the form

$$G = AB = AK = BK, A, B \subseteq G, K \triangleleft G.$$

Trifactorized Chernikov groups.

<u>**Problem**</u> **13.27 of Kourovka Note Book.**

Let $G = AB = AC = BC$ **be trifactorized.**

Is G **a Chernikov group, if** A**,** B **and** C **are Chernikov groups?**

<u>**Special case.**</u>

Is the group G a Chernikov group, if A, B, C have are Chernikov groups with $A/J(A)$, $B/J(B)$ and $C/J(C)$ **of index at most** 2?

(This seems to be open even for a locally finite group G)

Related problems.

Two Conjectures.

If A, B, C have Min, then G in general does not have Min.

If A, B, C are locally finite, then G in general is not locally finite.

A generalization of the Kegel-Wielandt-Theorem

Theorem (B.A., L.Kazarin, J. Alg. 311 (2007) 69-95).

Let the finite group $G = AB$ be the product of a nilpotent subgroup A and a subgroup B, then the normal closure of the center Z of B is a soluble normal subgroup of G.

In particular, if Z is non-trivial, then G contains a non-trivial abelian normal subgroup.

Question. Does this still hold when A contains a nilpotent subgroup of index at most 2?

Groups of polynomial growth

Definition. A finitely generated group has **polynomial growth** if the number of elements of length at most n (with respect to a symmetric generating set) is bounded by polynomial function $p(n)$.

Question: What can be said about groups $G = AB$ where A and B are finitely generated of polynomial growth?

Theorem (M. Gromov 1981)

A finitely generated group has polynomial growth if and only if it is nilpotent-by-finite.

There exist polycylic groups which are the product of a finitely generated abelian group and an infinite cyclic group.

ABA-groups.

Definition.

A group G with subgroups A and B is called an **ABA-group** or G has an **ABA-factorization**, if every element $g \in G$ can be represented in the form $g = aba_1$, where $a, a_1 \in A$, $b \in B$.

Clearly, a special case of this is a factorization of the form $G = AB$.

ABA-factorizations of finite groups were for instance studied by D. Gorenstein and I.N.Herstein, M.Guterman, L. Kazarin, Ya.Sysak, E.Vdovin, H.Alavi, C.Praeger, and others.

Remark on the inheritance of ABA-Factorizations

Lemma.

Let $G = ABA$ be a group with subgroups A and B. Then every subgroup H of G containing A can be represented in the form $H = A(B \cap H)A$.
In particular, if G is finite, then $|G| \leq |A|^2|B|$.

If N is a normal subgroup of G, then $G/N = \bar{A}\bar{B}\bar{A}$, where $\bar{A} = AN/N, \bar{B} = BN/N$.

Examples

1. Every 2-**transitive permutation group** is an ABA-group where A the stabilizer of a point and B a subgroup not contained in A.

2. Let G be a **simple group of Lie type over a field of characteristic** p, and let U be a Sylow p-subgroup of G. Then $B = N_G(U) = UH$ is the Borel subgroup of G and H its Cartan subgroup. Furthermore $N \leq N_G(H)$ and $W \simeq N/H$ is the Weyl group of G.
   Thus $\underline{G = BNB = UNU}$.

Symmetric and alternating groups

1. If A is a Sylow 5-subgroup of the **alternating group** $G = A_5$ and B is a Sylow 2-subgroup of G, then
   $G = ABA$, where $\underline{A \text{ is cyclic and } B \text{ is abelian}}$.

2. The **alternating group** $G = A_6$ **has an** ABA**-factorization**, where
   $\underline{A \text{ is a Sylow 3-subgroup of } G \text{ and } B \text{ is dihedral of order } 8}$.

3. The **symmetric group** $G = S_6$ **has an** ABA**-factorization**, where
   $\underline{A \text{ is a Sylow 2-subgroup of } G \text{ and } B \text{ is dihedral of order } 8}$.

Finite ABA-groups with A abelian and B cyclic

D.Gorenstein and I.N.Herstein showed that a finite ABA-group with cyclic subgroups A and B is soluble.

D.L.Zagorin and L.Kazarin announced in 1996 that a finite simple ABA-group with abelian subgroups A and B is isomorphic to a linear group $L_2(q)$ with even q. They also noted that in the case, **when A is abelian and B is cyclic, the group $G = ABA$ is soluble.**

The proof of these results used the Classification of all finite simple groups and was also obtained under this condition by E.P.Vdovin.

Two Solubility criteria for Finite ABA-groups

A proof of the following two theorems are contained in the Proceedings of the Ekaterinburg Group Theory Conference, May 2012 (in honor of S.N.Chernikov).

Theorem 6 (B.A., L.Kazarin).

Let G be a finite ABA-group with cyclic subgroup B. Then G is soluble in the following two cases:

a) A **abelian.**

b) A **nilpotent of odd order and** $(|A|, |B|) = 1$.

An example of a finite non-soluble ABA-group with A nilpotent and B cyclic.

Let A be a Sylow 2-subgroup of **the symmetric group** $G = S_5$, contained in the subgroup $A_1 = S_4$ of order 24. This subgroup is the stabilizer of a point 5 in S_5.
Since G is a 2-transitive permutation group, $G = A_1 B A_1$ for every subgroup B, not contained in A_1.
If B is the subgroup generated by the element b of the form $(1, 2, 3)(4, 5)$ then the order of $B = \langle b \rangle$ is 6.
Since the product AC of the subgroup $C = \langle (1, 2, 3) \rangle$ and 2-subgroup A of S_4 is A_1, we have $\underline{G = A_1 B A_1 = ACBCB = ABA}$.

Thus S_5 is a non-soluble ABA-group with A nilpotent and B cyclic.