On Deza Circulants

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G2A2, Yekaterinburg, 2015

Notations

We consider undirected graphs without loops and multiple edges.

For a graph Γ and its vertex x, define the neighbourhood of x:

$$\Gamma(x) := \{y \mid y \in V(\Gamma), y \sim x\}.$$

A graph Γ is called regular of valency k if

$$|\Gamma(x)| = k$$

holds for all $x \in \Gamma$.

Two definitions

A graph Δ is called a Deza graph with parameters (v, k, b, a) (usually $a < b$), if Δ has v vertices, and for any pair of vertices $x, y \in \Delta$:

$$|\Delta(x) \cap \Delta(y)| = \begin{cases} k, & \text{if } x = y, \\ a \text{ or } b, & \text{if } x \neq y. \end{cases}$$

$$\bigcup$$

A graph Γ is called strongly regular with parameters (v, k, λ, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$:

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y, \\ \lambda, & \text{if } x \sim y, \\ \mu, & \text{if } x \neq y \text{ and } x \nsim y. \end{cases}$$

A Deza graph Δ is called a strictly Deza graph, if Δ has diameter 2, and is not SRG.

One more

Let G be a finite group.
Let $S \subset G$ be a nonempty subset with the following properties

- $1_G \notin S$;
- $\forall s \in S \Rightarrow s^{-1} \in S$.

A graph $Cay(G, S)$ whose vertices are the elements of G, and the adjacency is defined by the following rule

$$x \sim y \Leftrightarrow xy^{-1} \in S, \quad \forall x, y \in G$$

is called a Cayley graph of group G with the connection set S.

A Cayley graph of a cyclic group is called a circulant.

In what follows, we consider a cyclic group of order n as $(\mathbb{Z}_n, +)$.

Problem and some results

Classification of some special classes of graphs that are also Cayley graphs is a popular problem between graph and group theories:

- Strongly regular circulants

  (Wielandt 1935; Bridges, Mena 1979; Hughes, van Lint, Wilson 1979; Ma 1984)
- Distance-regular circulants

  (Miklavic, Potocnik 2003)
- Strongly regular Cayley graphs of $C_{p^n} \times C_{p^n}$, p is a prime

  (Leifman, Muzychuk, 2005)
- Distance-regular Cayley graphs of dihedral groups

  (Miklavic, Potocnik, 2007)

Deza graphs on small number of vertices

(v, k, b, a)	Cayley
(8,4,2,0)	
(8,4,2,1)	+
(8,5,4,2)	+
(9,4,2,1)	+
(9,4,2,1)	
(10,5,4,2)	+
(12,5,2,1)	+
(12,6,3,2)	+
(12,6,3,2)	
(12,7,4,3)	+
(12,7,6,2)	+
(12,9,8,6)	
(13,8,5,4)	+

Erickson et al. (1999)

(v, k, b, a)	Cayley
(14,9,6,4)	+
(15,6,3,1)	
(16,5,2,1)	+
(16,7,4,2)	+
(16,7,4,2)	+
(16,8,4,2)	+
(16,9,6,4)	+
(16,9,6,4)	
(16,9,8,2)	+
(16,11,8,6)	+
(16,12,10,8)	+
(16,13,12,10)	+

Goryainov, Shalaginov (2011)

Cayley-Deza graph

Let G be a finite group, $|G| = v$, S be a connection set, $|S| = k$.

A Cayley graph $Cay(G, S)$ is a Deza graph (C.-D. graph) with parameters (v, k, b, a)

$$\Updownarrow$$

there are integers $b > a > 0$ and a partition $G = \{e\} \cup A \cup B$, such that

$$\text{the multiset } SS^{-1} = \{s_1 s_2^{-1} | s_1, s_2 \in S\} = k \cdot 1_G \cup a \cdot A \cup b \cdot B$$

Example

Let $G = C_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $S = \{1, 3, 4, 5, 7\}$.
Then $Cay(G, S)$ is a Deza graph with parameters $(8, 5, 4, 2)$ and with $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$ and

$$\text{the multiset } SS^{-1} = 5 \cdot 1_G \cup 2 \cdot A \cup 4 \cdot B$$

Cayley-Deza graphs over cyclic groups of order ≤ 95

All* Deza circulants found can be divided into several families:

1. $K_x[yK_2] \cong Cay(C_{2xy}, S_1)$ with $S_1 = (\mathbb{Z}_x \setminus \{0\}) \cup \{xy\}$;
2. $K_n \times K_4 \cong Cay(C_{4n}, S_2)$ with $S_2 = \{n, 2n, 3n\} \cup 4\mathbb{Z}_n \setminus \{0\}$;
3. divisible design graphs from a regular graphical Hadamard 4×4-matrix;
4. $Paley(p)[K_2]$, p is prime, $p \equiv 1(4)$;
5. from cyclotomic schemes with 3 classes, on p vertices, p is prime.
6. a new family of Deza graphs on pq vertices with $q - p = 4$, p, q are prime.

*: and two exceptions, $(8, 4, 2, 1)$, $(9, 4, 2, 1)$.

Problem

Show that any Deza circulant belongs to one of these families.

Divisible design graphs

Divisible design graph Γ with parameters $(v, k, \lambda_1, \lambda_2, m, n)$:

- a k-regular graph on $v = mn$ vertices
- its vertex set can be partitioned into m classes of size n
- any two distinct vertices $x, y \in \Gamma$ have exactly
  - λ_1 common neighbours, if x, y are from the same class
  - λ_2 common neighbours, if x, y are from the different classes
- proper if $m > 1$, $n > 1$ or $\lambda_1 \neq \lambda_2$

Note that any proper DDG Γ is a Deza graph
(unless $\Gamma \cong mK_n$ or $\overline{\Gamma} \cong mK_n$).

Regular graphical Hadamard matrices

Let H be an $m \times m$ Hadamard matrix.

H is called

- graphical if H is symmetric with constant diagonal,
- regular if all row and column sums are equal (to ℓ say).

The smallest regular graphical Hadamard matrices:

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

(3) Divisible design graphs from Hadamard matrices

DDG from H (Haemers, Kharaghani, Meulenberg, 2011)

Let H be a regular graphical Hadamard matrix of order $m \geq 4$ and row sum $\ell = \pm\sqrt{m}$. Let $n \geq 2$. Replace each entry of H

- with value -1 by $J_n - I_n$,
- with value $+1$ by I_n.

The result is the adjacency matrix of a DDG with parameters $(mn, n(m-\ell)/2 + \ell, (n-2)(m-\ell)/2, n(m-2\ell)/4 + \ell, m, n)$.

The smallest regular graphical Hadamard matrices:

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$\swarrow$ $\searrow$

(2) $K_n \times K_4$ (3) another family of Cayley-Deza graphs

(4) Deza circulant $Paley(p)[K_2]$

Let q be a prime power, $q \equiv 1(4)$. Define $S = \{x^2 \mid x \in \mathbb{F}_q^*\}$.

$$Paley(q) = Cay(\mathbb{F}_q^+, S).$$

It is an SRG with parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$.

Theorem (Wielandt, 1935)

A strongly regular circulant is $Paley(p)$, p is prime.

$Paley(q)[K_2]$ is a Deza graph with parameters

$$(2q, q, q-1, \frac{1}{2}(q-1)),$$

and it is a circulant if and only if q is prime (not prime power).

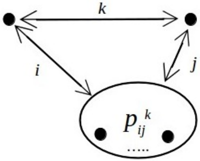
Theorem

Let p be a prime, and Δ be a Cayley-Deza graph over C_{2p}.
Then $p \equiv 1(4)$ and $\Delta \cong Paley(p)[K_2]$.

Association schemes $\mathcal{S} = (V, \mathcal{R})$

V — a set of v elements, $\mathcal{R}$ — a partition of $V \times V$ into $d+1$ binary relations $R_0, R_1, \ldots, R_d$, which satisfy:

- $R_0 = \{(x,x) \mid x \in V\}$, the identity relation,
- $\forall i$: $R_i^\top = \{(y,x) \mid (x,y) \in R_i\}$ is a member of $\mathcal{R}$,
- if $(x,y) \in R_k$, then the number of z such that



$(x,z) \in R_i$ $(z,y) \in R_j$

  is a constant denoted by p_{ij}^k.

An association scheme $\mathcal{S}$ is

- commutative if $p_{ij}^k = p_{ji}^k$, for $\forall\ i,j,k$.

  $\cup$

- symmetric if $R_i = R_i^\top$, for $\forall\ i$.

Cyclotomic scheme

- Let q be a prime power, and e be a divisor of $q-1$.
- Fix a primitive element α of the multiplicative group of $\mathbb{F}_q$.
- $\langle \alpha^e \rangle$ is a subgroup of $\mathbb{F}_q^*$ of index e and its cosets are $\alpha^i \langle \alpha^e \rangle$, $(0 \le i \le e-1)$.

Define $R_0 = \{(x,x) \mid x \in \mathbb{F}_q\}$ and

$$R_i = \{(x,y) \mid x-y \in \alpha^i \langle \alpha^e \rangle, x,y \in \mathbb{F}_q\} \quad (1 \le i \le e).$$

Then $(V,\mathcal{R}) = (\mathbb{F}_q, \{R_i\}_{i=0}^{e})$ forms an association scheme and it is called the cyclotomic scheme of class e on $\mathbb{F}_q$.

The cyclotomic scheme of class e on $\mathbb{F}_q$ is symmetric if and only if q or $(q-1)/e$ is even.

Cyclotomic scheme on 3 classes

Let q be an odd prime, and $e = 3$ be a divisor of $q - 1$.
Let $\mathcal{S}$ be the cyclotomic scheme of class 3 on $\mathbb{F}_q$.
Mathon (1975) calculated the parameters p_{ij}^k of $\mathcal{S}$:

$$\begin{array}{c} p_{ij}^1 \\ 1 \\ 2 \\ 3 \end{array} \quad \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} t-1 & s & r \\ s & r & t \\ r & t & s \end{bmatrix} \end{array}, \quad \begin{array}{c} p_{ij}^2 \\ 1 \\ 2 \\ 3 \end{array} \quad \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} s & r & t \\ r & t-1 & s \\ t & s & r \end{bmatrix} \end{array}$$

$$\begin{array}{c} p_{ij}^3 \\ 1 \\ 2 \\ 3 \end{array} \quad \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} r & t & s \\ t & s & r \\ s & r & t-1 \end{bmatrix} \end{array},$$

where $t = e - r - s$, and r, s, t satisfy additional equation:

$$1 + 2(r + s) - 3(r - s)^2 = (1 + 3(r + s) - 2e)^2.$$

(5) Deza circulant from cyclotomic schemes

Recall the criterion to form a Deza graph from $\mathcal{S}$.

Theorem (Erickson et al., 1999)

For a subset $F \subset \{1, 2, \ldots, d\}$, a graph Γ with adjacency matrix $A_F = \sum_{f \in F} A_f$ is a Deza graph if and only if

$$\sum_{f,g \in F} p^k_{fg}$$

takes on at most two values, as k ranges over $\{1, 2, \ldots, d\}$.

In our case, we may assume that $F = \{1\}$ or $F = \{1, 2\}$.

Deza $\Leftrightarrow \#\{\blacksquare, \blacksquare, \blacksquare\} = 2$

$$F = \{1\} \quad \begin{array}{c} p^1_{ij} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} t-1 & s & r \\ s & r & t \\ r & t & s \end{bmatrix} \end{array}, \quad \begin{array}{c} p^2_{ij} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} s & r & t \\ r & t-1 & s \\ t & s & r \end{bmatrix} \end{array}$$

$$\begin{array}{c} p^3_{ij} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} r & t & s \\ t & s & r \\ s & r & t-1 \end{bmatrix} \end{array}$$

$$F = \{1, 2\} \quad \begin{array}{c} p^1_{ij} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} t-1 & s & r \\ s & r & t \\ r & t & s \end{bmatrix} \end{array}, \quad \begin{array}{c} p^2_{ij} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} s & r & t \\ r & t-1 & s \\ t & s & r \end{bmatrix} \end{array}$$

$$\begin{array}{c} p^3_{ij} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} r & t & s \\ t & s & r \\ s & r & t-1 \end{bmatrix} \end{array}$$

(5) Deza graphs from cyclotomic schemes

Theorem

Let q be a prime power, and $\mathcal{S}$ be the cyclotomic scheme of class 3 on $\mathbb{F}_q$. Let $F \subset \{1, 2, 3\}$.

Then a graph with adjacency matrix $A_F = \sum_{f \in F} A_f$ is a Deza if and only if q is a prime and one of the following holds:

- $|F| = 1$ and $q = x^2 + 3$ for some integer x,
- $|F| = 2$ and $q = x^2 + 12$ for some integer x.

Conjecture (Bunyakovsky, 1857)

Let $f(x)$ be a polynomial in one variable satisfying:

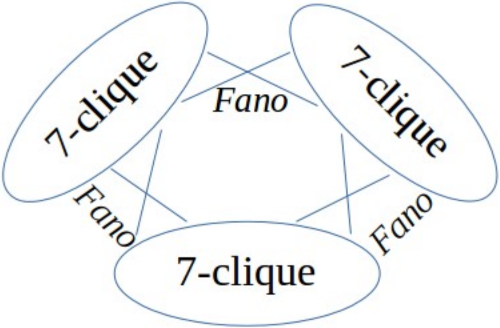
- the leading coefficient of $f(x)$ is positive,
- the polynomial is irreducible over the integers,
- the coefficients of $f(x)$ are relatively prime.

Then $f(n)$ is prime for infinitely many positive integers n.

New family (6)

Among all Cayley-Deza graphs on ≤ 95 vertices, we found two examples with parameters $(21, 12, 7, 6)$ and $(77, 40, 21, 20)$. These parameters sets did not satisfy any previously known construction.

The graph on 21 vertices has two systems of imprimitivity $(3 \times 7$ and $7 \times 3)$, one of which gives the following picture.



New family (6)

We first described a connection set S for the graph on 21 vertices, and then generalized it.

Let p, q be two prime powers with $q - p = 4$, $p \equiv 3(4)$.

Define

- $S_p := \{x^2 \mid x \in \mathbb{F}_p^*\}$ and, similarly, S_q,
- $\overline{S}_p = \mathbb{F}_p^* \setminus S_p$ and, similarly, $\overline{S}_q$.

Let

$$S_0 = \{(x, 0) \mid x \in \mathbb{F}_q^*\},$$
$$S_1 = S_q \times \overline{S}_p, \quad S_2 = \overline{S}_q \times S_p.$$

Theorem (Joint with Galina Isakova)

$Cay(\mathbb{F}_q^+ \times \mathbb{F}_p^+, S_0 \cup S_1 \cup S_2)$ is a Deza graph with parameters $(v, \frac{1}{2}(v+3), \frac{1}{4}(v+7), \frac{1}{4}(v+3))$, $v = pq$.

If p and q are prime it is a circulant because $\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \times \mathbb{Z}_q$

It is only conjectured that there exist infinitely many such pairs of prime numbers p, q.

Strictly Deza circulants having ≤ 95 vertices

C8	■(8,4,2,1)■ (8,5,4,2)
C9	■(9,4,2,1)■
C10	(10,5,4,2)
C12	(12,5,2,1) (12,7,4,3) (12,7,6,2) (12,9,8,6)
C13	(13,8,5,4)
C16	(16,9,8,2) (16,13,12,10)
C18	(18,13,12,8)
C19	(19,6,2,1)
C20	(20,7,3,2) (20,11,10,2) (20,13,9,8) (20,17,16,14)
C21	(21,12,7,6)
C24	(24,13,12,2) (24,17,16,10) (24,19,18,14) (24,21,20,18)
C26	(26,13,12,6)
C28	(28,9,5,2) (28,15,14,2) (28,19,15,12) (28,25,24,22)
C30	(30,21,20,12) (30,25,24,20)
C32	(32,17,16,2) (32,25,24,18) (32,29,28,26)
C34	(34,17,16,8)
C36	(36,11,7,2) (36,19,18,2) (36,25,21,16) (36,25,24,14) (36,31,30,26) (36,33,32,30)
C37	(37,24,16,15)
C40	(40,21,20,2) (40,31,30,22) (40,33,32,26) (40,37,36,34)
C42	(42,29,28,16) (42,37,36,32)
C44	(44,13,9,2) (44,23,22,2) (44,31,27,20) (44,41,40,38)
C48	(48,25,24,2) (48,33,32,18) (48,37,36,26) (48,41,40,34) (48,43,42,38) (48,45,44,42)

$K_x[yK_2]$, $K_n \times K_4$, Divisible design graphs, $Paley(p)[K_2]$, From cyclotomic schemes with 3 classes, New family, ■ — ?Sporadic?

Strictly Deza circulants having ≤ 95 vertices

C50	(50,41,40,32)
C52	(52,15,11,2) (52,27,26,2) (52,37,33,24) (52,49,48,46)
C54	(54,37,36,20) (54,49,48,44)
C56	(56,29,28,2) (56,43,42,30) (56,49,48,42) (56,53,52,50)
C58	(58,29,28,14)
C60	(60,17,13,2) (60,31,30,2) (60,41,40,22) (60,43,39,28) (60,49,48,38) (60,51,50,42) (60,55,54,50) (60,57,56,54)
C61	(61,40,27,24)
C64	(64,49,48,34) (64,57,56,50) (64,61,60,58)
C66	(66,45,44,24) (66,61,60,56)
C67	(67,22,9,6)
C68	(68,19,15,2) (68,35,34,2) (68,49,45,32) (68,65,64,62)
C70	(70,57,56,44) (70,61,60,52)
C72	(72,37,36,2) (72,49,48,26) (72,55,54,38) (72,61,60,50) (72,65,64,58) (72,67,66,62) (72,69,68,66)
C74	(74,37,36,18)
C76	(76,21,17,2) (76,39,38,2) (76,55,51,36) (76,73,72,70)
C77	(77,40,21,20)
C78	(78,53,52,28) (78,73,72,68)
C80	(80,41,40,2) (80,61,60,42) (80,65,64,50) (80,71,70,62) (80,73,72,66) (80,77,76,74)
C82	(82,41,40,20)
C84	(84,23,19,2) (84,43,42,2) (84,57,56,30) (84,61,57,40)
C88	(88,67,66,46) (88,81,80,74) (88,85,84,82)
C90	(90,61,60,32) (90,73,72,56) (90,81,80,72) (90,85,84,80)
C92	(92,25,21,2) (92,47,46,2) (92,67,63,44) (92,89,88,86)

$K_x[yK_2]$, $K_n \times K_4$, Divisible design graphs, $Paley(p)[K_2]$, From cyclotomic schemes with 3 classes, New family, ■ — ?Sporadic?

Summary

Results:

- It seems that there are 6 families of Deza circulants (with two exceptions on 8 and 9 vertices):
  - $K_x[yK_2]$;
  - $K_n \times K_4$;
  - from DDG based on the Hadamard 4×4-matrix;
  - $Paley(p)[K_2]$;
  - from cyclotomic schemes with 3 classes;
  - a new family of Deza graphs on pq vertices with $q - p = 4$.
  - two exceptions on 8 and 9 vertices.
- We can completely characterize Deza circulants over C_{2p}.

Some questions:

- generalize the result for C_{2p} to C_{4p};
- If p or q is not prime are there circulants in the new family (6)? (in this case $\mathbb{F}_q^+ \times \mathbb{F}_p^+$ is not cyclic).

Thank you!