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Finite dimensional irreducible representations of the tridiagonal algebras

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plan

Definition of the tridiagonal algebra (TD-alg.)

standardization, type I, II, III

Motivation : Terwilliger alg. of (P & Q)-poly. schemes

Augmented tridiagonal algebra (aug. TD-alg.)

$$\underset{\substack{\text{TD-alg.}\\ \text{standard}}}{\mathcal{A}} \hookrightarrow \underset{\text{aug. TD-alg.}}{\mathcal{T}}$$

Construction of f.d. representations of $\mathcal{T}$

Drinfeld polynomials $P_V(x)$ for a $\mathcal{T}$-module V

Main Theorems : classification of f.d. irred. rep. of $\mathcal{T}$, $\mathcal{A}$.

Notation

$[X, Y] = XY - YX$ Lie bracket

$[X, Y]_q = qXY - q^{-1}YX$ q- bracket

$[i] = [i]_q = \dfrac{q^i - q^{-i}}{q - q^{-1}}$ q- integer

$[i]! = [i][i-1] \cdots [1]$ q- factorial

TD-algebra (tridiagonal algebra)

$A = \langle Z, Z^* \rangle$

associative $\mathbb{C}$-algebra with 1

$$\begin{cases} Z^3 Z^* - (\beta+1)(Z^2 Z^* Z - Z Z^* Z^2) - Z^* Z^3 = \gamma [Z^2, Z^*] + \delta [Z, Z^*] \\ Z^{*3} Z - (\beta+1)(Z^{*2} Z Z^* - Z^* Z Z^{*2}) - Z Z^{*3} = \gamma^* [Z^{*2}, Z] + \delta^* [Z^*, Z] \end{cases}$$

$\beta = q^2 + q^{-2}$ invariant under affine transformations

$$Z \longmapsto \lambda Z + \mu \qquad (\lambda, \mu \in \mathbb{C},\ \lambda \neq 0)$$

$$Z^* \longmapsto \lambda^* Z + \mu^* \qquad (\lambda^*, \mu^* \in \mathbb{C},\ \lambda^* \neq 0)$$

Standardization (standardized TD-alg.)

$$\mathcal{A} = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)} = \langle z, z^* \rangle$$

associative $\mathbb{C}$-algebra with 1

$q \in \mathbb{C}^\times$, $(\varepsilon, \varepsilon^*) = (1,1), (1,0), (0,0)$

type I: $q^2 \neq \pm 1$

$$\begin{cases} z^3 z^* - [3]_q z^2 z^* z + [3]_q z z^* z^2 - z^* z^3 = -\varepsilon (q^2 - q^{-2})^2 [z, z^*] \\ z^{*3} z - [3]_q z^{*2} z z^* + [3]_q z^* z z^{*2} - z z^{*3} = -\varepsilon^* (q^2 - q^{-2})^2 [z^*, z] \end{cases}$$

type II: $q^2 = 1$

$$\begin{cases} z^3 z^* - 3 z^2 z^* z + 3 z z^* z^2 - z^* z^3 = 2\varepsilon [z^2, z^*] + (1-\varepsilon)[z, z^*] \\ z^{*3} z - 3 z^{*2} z z^* + 3 z^* z z^{*2} - z z^{*3} = 2\varepsilon^* [z^{*2}, z] + (1-\varepsilon^*)[z^*, z] \end{cases}$$

type III: $q^2 = -1$

$$\begin{cases} z^3 z^* + z^2 z^* z - z z^* z^2 - z^* z^3 = 4 [z, z^*] \\ z^{*3} z + z^{*2} z z^* - z^* z z^{*2} - z z^{*3} = 4 [z^*, z] \end{cases}$$

Examples

type I, $\varepsilon = \varepsilon^* = 0$

$$\begin{cases} Z^3 Z^* - [3]_q Z^2 Z^* Z + [3]_q Z Z^* Z^2 - Z^* Z^3 = 0 \\ Z^{*3} Z - [3]_q Z^{*2} Z Z^* + [3]_q Z^* Z Z^{*2} - Z Z^{*3} = 0 \end{cases}$$

q- Serre relations

$\mathcal{A} = \langle Z, Z^* \rangle \simeq$ the positive part of $U_q(\widehat{sl_2})$

type II, $\varepsilon = \varepsilon^* = 0$

$$\begin{cases} Z^3 Z^* - 3 Z^2 Z^* Z + 3 Z Z^* Z^2 - Z^* Z^3 = [Z, Z^*] \\ Z^{*3} Z - 3 Z^{*2} Z Z^* + 3 Z^* Z Z^{*2} - Z Z^{*3} = [Z^*, Z] \end{cases}$$

Dolan - Grady relations

$\mathcal{A} = \langle Z, Z^* \rangle \simeq$ the universal enveloping algebra of the Onsager algebra

type I, $\varepsilon = \varepsilon^* = 1$

$$\begin{cases} Z Z^* - [3]_q Z^2 Z^* Z + [3]_q Z Z^* Z^2 - Z^* Z^3 = -(q^2 - q^{-2})^2 [Z, Z^*] \\ Z^{*3} Z - [3]_q Z^{*2} Z Z^* + [3]_q Z^* Z Z^{*2} - Z Z^{*3} = -(q^2 - q^{-2})^2 [Z^*, Z] \end{cases}$$

$\mathcal{A}$: <u>the q-Onsager algebra</u>

(P and Q)-polynomial association schemes

$\|$

Discrete analogue of
compact symmetric spaces of rank 1

Terwilliger algebra of a (P & Q)-poly. assoc. scheme

$$\langle A, A^* \rangle \quad \supset \quad \langle A \rangle$$

Terwilliger alg. — Bose-Mesner alg.

non commutative — commutative

semi simple

finite dimensional

V : standard module

$\cup$

W : irreducible $\langle A, A^* \rangle$-module

<u>TD-pair</u> $A|_W,\ A^*|_W \in End(W)$

tridiagonal pair

V : finite dimensional vector space over $\mathbb{C}$

$A, A^* \in \mathrm{End}(V)$ <u>TD-pair</u>

1) A, A^* are diagonalizable on V

$$V = \bigoplus_{i=0}^{d} V_i \quad \text{eigenspace decomposition of } A$$

$$V = \bigoplus_{i=0}^{d^*} V_i \quad \text{eigenspace decomposition of } A^*$$

2) $A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d^*$

where $V_{-1}^* = V_{d^*+1}^* = 0$

3) $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d$

where $V_{-1} = V_{d+1} = 0$

4) V has no proper subspace invariant under A and A^*.

<u>Problem</u> Classify TD-pairs up to isomorphism.

TD-pairs A, A $\longleftarrow\text{-----}\longrightarrow$ finite dimensional irreducible representations of TD-algebra $\mathcal{A}$ with $(*)$

$$\rho : \mathcal{A} = \langle z, z^* \rangle \longrightarrow \mathrm{End}(V)$$
$$z, z^* \longmapsto A, A^*$$

irreducible representation of $\mathcal{A}$
such that
$(*)$ $\rho(z), \rho(z^*)$ are diagonalizable

Remark

Need a careful treatment
in the case of
$q^2 \neq \pm 1$ and q is a root of unity.

Problem Classify finite dimensional irreducible representations of $\mathcal{A}$ with $(*)$.

Assumption For simplicity, we assume
q is not a root of unity
in the case of type I.

<u>aug. TD-alg.</u> (augmented tridiagonal algebra)

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ associative $\mathbb{C}$-algebra with 1

$q \in \mathbb{C}^\times$, $(\varepsilon,\varepsilon^*) = (1,1), (1,0), (0,0)$

type I : $q^2 \neq \pm 1$, $\mathcal{T} = \langle x, y, k^{\pm 1} \rangle$

$$kk^{-1} = k^{-1}k = 1$$

$$kxk^{-1} = q^2 x, \quad kyk^{-1} = q^{-2} y$$

$$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = \delta x(\varepsilon^* k^2 - \varepsilon k^{-2})x$$

$$y^3x - [3]_q y^2xy + [3]_q yxy^2 - xy^3 = -\delta y(\varepsilon^* k^2 - \varepsilon k^{-2})y$$

where $\delta = -(q-q^{-1})(q^2-q^{-2})(q^3-q^{-3})$

type II : $q^2 = 1$, $\mathcal{T} = \langle x, y, k \rangle$

$$kx - xk = 2x, \quad ky - yk = -2y$$

$$x^3y - 3x^2yx + 3xyx^2 - yx^3 = \begin{cases} -12xkx & \text{if } (\varepsilon,\varepsilon^*) = (1,1) \\ -6x^2 & (1,0) \\ 0 & (0,0) \end{cases}$$

$$y^3x - 3y^2xy + 3yxy^2 - xy^3 = \begin{cases} 12yky & \text{if } (\varepsilon,\varepsilon^*) = (1,1) \\ 6y^2 & (1,0) \\ 0 & (0,0) \end{cases}$$

type III : $q^2 = -1$,

$$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)} = \langle x, y, k, \tau \rangle$$

$$\tau^2 = 1$$

$$\tau k - k\tau = 0, \quad \tau x + x\tau = 0, \quad \tau y + y\tau = 0$$

$$kx - xk = 2x, \quad ky - yk = -2y$$

$$x^3 y + x^2 y x - x y x^2 - y x^3 = 16 x k x$$

$$y^3 x + y^2 x y - y x y^2 - x y^3 = -16 y k y$$

Proposition

$$\iota_t : \mathcal{A} \longrightarrow \mathcal{T} \qquad \text{alg. homomorphism}$$
$$Z, Z^* \longmapsto Z_t, Z_t^*$$

where

type I : $t \in \mathbb{C}$ $(t \neq 0)$

$$Z_t = x + tk + \varepsilon t^{-1} k^{-1}$$
$$Z_t^* = y + \varepsilon^* t^{-1} k + t k^{-1}$$

type II : $t \in \mathbb{C}$

$$Z_t = x + \frac{k+t-1}{2}\left(\varepsilon \frac{k+t-1}{2} + 1\right)$$
$$Z_t^* = y + \frac{k-t-1}{2}\left(\varepsilon^* \frac{k-t-1}{2} + 1\right)$$

type III : $t \in \mathbb{C}$

$$Z_t = x + \tau(k+t)$$
$$Z_t^* = y + \tau(k-t)$$

Remark ι_t is injective.

Proposition

$\rho_0 : \mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)} \longrightarrow End(V)$ f.d. irred. rep.

such that $(*)$ $\rho_0(z)$, $\rho_0(z^*)$ are diagonalizable.

Then

$\exists\, \iota_t : \mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)} \longrightarrow \mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$

and

$\exists\, \rho_1 : \mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)} \longrightarrow End(V)$ f.d. irred. rep.

s.t.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho_0"] \arrow[d, "\iota_t"'] \arrow[dr, phantom, "\circlearrowleft"] & End(V) \\ \mathcal{T} \arrow[ur, "\rho_1"'] & \end{tikzcd}$$

Remark Recall we assumed that q is not a root of unity in the case of type I, i.e., $q^2 \neq \pm 1$.

If q is a root of unity and $q^2 \neq \pm 1$, we need a careful treatment. (Need to assume ρ_1 is regular.)

Problem

(1) Construct, up to isomorphism,
all the finite-dimensional irreducible representations

$$\rho_1 : \mathcal{T} \longrightarrow \mathrm{End}(V).$$

(2) For a f.d. irred. rep. ρ_1 of $\mathcal{T}$ constructed in (1) above,
determine when the representation of $\mathcal{A}$

$$\rho_0 = \rho_1 \circ \iota_t : \quad \mathcal{A} \xrightarrow{\iota_t} \mathcal{T} \xrightarrow{\rho_1} \mathrm{End}(V)$$

is irreducible.

(3) Determine the isomorphism class of
the $\mathcal{A}$-module V via ρ_0 constructed in (2) above.

V : finite dimensional irreducible $\mathcal{T}$-module

Lemma

k is diagonalizable on V.

$$V = \bigoplus_{i=0}^{d} U_i \qquad \text{e.s. decomp. of } k \ (\underline{\text{weight space decomp.}})$$

$$k\big|_{U_i} = \begin{cases} s q^{2i-d} & \text{for type I} \\ s + 2i - d & \text{for type II, III} \end{cases}$$

x, y are nilpotent on V

$$x U_i \subseteq U_{i+1}, \quad y U_i \subseteq U_{i-1}, \quad 0 \leq i \leq d$$

where $U_{-1} = U_{d+1} = 0$.

d is called the diameter.
s is called the type.

Remark In the case of type III,

$$\tau\big|_{U_i} = (-1)^i \quad \text{or} \quad (-1)^{i+1}, \quad 0 \leq i \leq d.$$

We may assume $\tau\big|_{U_i} = (-1)^i$ by means of the automorphism of $\mathcal{T}$

$$\tau \longmapsto -\tau, \qquad x, y, k \longmapsto x, y, k$$

Theorem $\dim U_0 = 1$

Theorem (Character formula)

$$\mathrm{ch}(\lambda) = \sum_{i=0}^{d} (\dim U_i)\, \lambda^i = \prod_{i=1}^{n} \frac{1-\lambda^{l_i+1}}{1-\lambda}$$

for some $l_1, l_2, \cdots, l_n \in \mathbb{Z}_{>0}$.

Remark Recall we assumed q is not a root of unity in the case of type I, i.e., $q^2 \neq \pm 1$.

If $q^2 \neq \pm 1$ and q is a root of unity, we need a careful treatment. (Need to assume the T-module V is regular.)

type I $\quad \mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)} = \langle x, y, k^{\pm 1} \rangle, \quad q^2 \neq \pm 1$

V : f. d. irred. $\mathcal{T}$-module of type s, diameter d

$$V = \bigoplus_{i=0}^{d} U_i, \qquad k|_{U_i} = s q^{2i-d}, \quad 0 \le i \le d$$

$$x U_i \subseteq U_{i+1}, \quad y U_i \subseteq U_{i-1}, \quad 0 \le i \le d$$

$$q^{2i} \neq 1, \quad 1 \le i \le d$$

exceptional case

$$q^{2(d+1)} = 1.$$

$$U_{-1} = U_d, \quad U_{d+1} = U_0$$

V is regular if type II or III or type I and

(i) non-circulant : $x U_d = 0, \quad y U_0 = 0$

(ii)
$$x^d y x - x y x^d = (q-q^{-1})^3 [d-1][d](\varepsilon^* s^2 - \varepsilon s^{-2}) x^d \quad \text{on } U_0$$
$$y^d x y - y x y^d = -(q-q^{-1})^3 [d-1][d](\varepsilon^* s^2 - \varepsilon s^{-2}) y^d \quad \text{on } U_d$$

Remark $\quad$ type I and non-exceptional case $\Rightarrow$ regular

$$\text{standard TD-pairs} \xleftrightarrow{\;1:1\;} \rho_0 : \underset{\displaystyle \mathcal{A}}{\overset{\displaystyle \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}}{\shortparallel}} \longrightarrow \mathrm{End}(V) \quad \text{f.d. irred. rep.}$$

s.t.

$\rho_0 = \rho_1 \circ \iota_t$ for some

$\iota_t : \mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)} \longrightarrow \mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$

and

$\rho_1 : \mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)} \longrightarrow \mathrm{End}(V)$

f.d. irred. rep. of $\mathcal{T}$

that is <u>regular</u>.

Construction of $\mathcal{T}$-modules

type I $q^2 \neq \pm 1$

$$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)} = \langle x, y, k^{\pm 1} \rangle$$

$V(l, a) = \langle v_0, v_1, \cdots, v_l \rangle$ evaluation module for $\mathcal{T}$

$l \in \mathbb{Z}_{>0}$, $a, s \in \mathbb{C}^\times$, $\dim V(l,a) = l+1$

$$k(s) v_i = s q^{2i-l} v_i, \quad 0 \le i \le d,$$

$$x(s) v_i = -(q - q^{-1})^2 [i+1] (as + \varepsilon s^{-1} q^{-2i+l-1}) v_{i+1}, \quad 0 \le i \le d$$

$$y(s) v_i = [l-i+1] (\varepsilon^* a^{-1} s q^{2i-l-1} + s^{-1}) v_{i-1}, \quad 0 \le i \le d$$

where $v_{-1} = v_{l+1} = 0$.

$$\mathcal{T} \longrightarrow V(l,a)$$

k, x, y act on $V(l,a)$ as $k(s), x(s), y(s)$.

diameter l, type s

Remark

In the case of $(\varepsilon, \varepsilon^*) = (1, 0)$,

$a = 0$ is allowed

with the understanding of $\varepsilon^* a^{-1} = 0$.

tensor product of evaluation modules

$$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$$

$$V(l_i, a_i) = \langle v_0^{(i)}, v_1^{(i)}, \cdots, v_{l_i}^{(i)} \rangle \quad \text{ev. module}$$

$$u_{j_1 \cdots j_n} = v_{j_1}^{(1)} \otimes \cdots \otimes v_{j_n}^{(n)}, \quad 0 \le j_1 \le l_1, \cdots, 0 \le j_n \le l_n$$

$$V = \bigoplus_{i=0}^{d} U_i, \qquad d = l_1 + \cdots + l_n$$

$$U_i = \langle u_{j_1 \cdots j_n} \mid j_1 + \cdots + j_n = i \rangle$$

$$\mathcal{T} \longrightarrow V$$

$$k\, u_{j_1 \cdots j_n} = s q^{2i-d} u_{j_1 \cdots j_n}, \qquad i = j_1 + \cdots + j_n$$

$$x\, u_{j_1 \cdots j_n} = \sum_{i=1}^{n} (1 \otimes \cdots \otimes x(s_i) \otimes 1 \otimes \cdots \otimes 1)\, u_{j_1 \cdots j_n}$$

$$y\, u_{j_1 \cdots j_n} = \sum_{i=1}^{n} (1 \otimes \cdots \otimes y(s_i) \otimes 1 \otimes \cdots \otimes 1)\, u_{j_1 \cdots j_n}$$

where $s_i = s\, q^{\sum_{\nu=1}^{i-1} (2 j_\nu - l_\nu)}$.

Remark V has types : diameter d

$$k|_{U_i} = s q^{2i-d}$$

$$x U_i \subseteq U_{i+1}, \quad y U_i \subseteq U_{i-1}$$

$$(U_{-1} = U_{d+1} = 0)$$

<u>Remark</u>

the $U_q(sl_2)$-loop algebra

$$\exists \varphi_s : \mathcal{T} \hookrightarrow U_q(L(sl_2)) \quad \text{embedding}$$

$$U_q(L(sl_2)) \xrightarrow{\text{act}} V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$$

type II $\quad q^2 = 1$

$$\mathcal{T} = \mathcal{T}_q^{(\varepsilon\varepsilon^*)} = \langle x, y, k \rangle$$

$V(\ell, a) = \langle v_0, v_1, \cdots, v_\ell \rangle$ evaluation module for $\mathcal{T}$

$\ell \in \mathbb{Z}_{>0}, \quad a, s \in \mathbb{C} \quad (a=0, s=0$ allowed$)$

$\dim V(\ell, a) = \ell + 1$

$$k(s) v_i = (2i - \ell + s) v_i$$

$$x(s) v_i = (i+1)\left(a + \varepsilon\left(s + \frac{2i-\ell+1}{2}\right)\right) v_{i+1}$$

$$y(s) v_i = (\ell - i + 1)\left(1 + \varepsilon^*\left(a - s - \frac{2i-\ell+1}{2}\right)\right) v_{i-1}$$

where $v_{-1} = v_{\ell+1} = 0$.

$$\mathcal{T} \longrightarrow V(\ell, a)$$

k, x, y act on $V(\ell, a)$ as $k(s), x(s), y(s)$.

diameter ℓ, type s

Construction type II

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tensor product of evaluation modules

$$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$$

$$V(l_i, a_i) = \langle v_0^{(i)}, v_1^{(i)}, \cdots, v_{l_i}^{(i)} \rangle \qquad \text{ev. module}$$

$$V = \bigoplus_{i=0}^{d} U_i , \qquad d = l_1 + \cdots + l_n$$

$$U_i = \langle u_{j_1 \cdots j_n} = v_{j_1}^{(1)} \otimes \cdots \otimes v_{j_n}^{(n)} \mid j_1 + \cdots + j_n = i \rangle$$

$$\mathfrak{T} \longrightarrow V$$

$$k\, u_{j_1 \cdots j_n} = (2i - d + s)\, u_{j_1 \cdots j_n}, \qquad i = j_1 + \cdots + j_n$$

$$x\, u_{j_1 \cdots j_n} = \sum_{i=1}^{n} (1 \otimes \cdots \otimes 1 \otimes x(s_i) \otimes 1 \otimes \cdots \otimes 1)\, u_{j_1 \cdots j_n}$$

$$y\, u_{j_1 \cdots j_n} = \sum_{i=1}^{n} (1 \otimes \cdots \otimes 1 \otimes y(s_i) \otimes 1 \otimes \cdots \otimes 1)\, u_{j_1 \cdots j_n}$$

where

$$s_i = s + \sum_{\nu=1}^{i-1} (2 j_\nu - l_\nu).$$

V has type s, diameter d:

$$k|_{U_i} = 2i - d + s$$

$$x\, U_i \subseteq U_{i+1}, \qquad y\, U_i \subseteq U_{i-1}$$

$$(U_{-1} = U_{d+1} = 0).$$

Question $q^2 = 1$

$$\mathcal{T} \hookrightarrow ?? \quad \text{embedding}$$

Hopf algebra

$$\downarrow$$

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

If $(\varepsilon, \varepsilon^*) = (0,0)$, then Yes.

$\mathcal{A} \simeq$ the universal enveloping algebra of the Onsager algebra

$$\mathcal{A} \xhookrightarrow{\iota_t} \mathcal{T} \xhookrightarrow{\varphi_s} L(sl_2)$$

the sl_2-loop algebra

type III $\quad q^2=-1$

$$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)} = \langle x, y, k, \tau \rangle$$

$V(l, a) = \langle v_0, v_1, \cdots, v_l \rangle$ evaluation module for $\mathcal{T}$

$l \in \mathbb{Z}_{>0}$, $a, s \in \mathbb{C}$ $\quad$ ($a=0$, $s=0$ allowed)

$\dim V(l,a) = l+1$

$$\tau(s) v_i = (-1)^i v_i$$

$$k(s) v_i = (2i - l + s) v_i$$

$$x(s) v_i = \begin{cases} 2(-1)^i (i+1) v_{i+1} & \text{if } i \text{ is odd,} \\ 2(-1)^i \left(a + s + \frac{2i-l+1}{2}\right) v_{i+1} & \text{if } i \text{ is even} \end{cases}$$

$$y(s) v_i = \begin{cases} 2(-1)^{i-1} (l-i+1) v_{i-1} & \text{if } l-i \text{ is odd,} \\ 2(-1)^{i-1} \left(a - s - \frac{2i-l-1}{2}\right) v_{i-1} & \text{if } l-i \text{ is even} \end{cases}$$

where $v_{-1} = v_{l+1} = 0$.

$$\mathcal{T} \longrightarrow V(l, a)$$

τ, k, x, y act on $V(l,a)$ as $\tau(s), k(s), x(s), y(s)$

diameter l, $\quad$ type s

tensor product of evaluation modules

$$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$$

$$V(l_i, a_i) = \langle v_0^{(i)}, v_1^{(i)}, \cdots, v_{l_i}^{(i)} \rangle \qquad \text{ev. module}$$

$$V = \bigoplus_{i=0}^{d} U_i, \qquad d = l_1 + \cdots + l_n$$

$$U_i = \langle u_{j_1 \cdots j_n} = v_{j_1}^{(1)} \otimes \cdots \otimes v_{j_n}^{(n)} \mid j_1 + \cdots + j_n = i \rangle$$

$$\mathcal{T} \longrightarrow V$$

$$\tau u_{j_1 \cdots j_n} = (-1)^i, \qquad i = j_1 + \cdots + j_n$$

$$k u_{j_1 \cdots j_n} = (2i - d + s) u_{j_1 \cdots j_n}, \qquad i = j_1 + \cdots + j_n$$

$$x u_{j_1 \cdots j_n} = \sum_{i=1}^{n} (\tau(s) \otimes \cdots \otimes \tau(s) \otimes x(s_i) \otimes 1 \otimes \cdots \otimes 1) u_{j_1 \cdots j_n}$$

$$y u_{j_1 \cdots j_n} = (-1)^d \sum_{i=1}^{n} (-1)^{\sum_{\nu=1}^{i} l_\nu} (\tau(s) \otimes \cdots \otimes \tau(s) \otimes y(s_i) \otimes 1 \otimes \cdots \otimes 1) u_{j_1 \cdots j_n}$$

where

$$s_i = s + \sum_{\nu=1}^{i-1} (2 j_\nu - l_\nu).$$

V has <u>type s</u>, diameter d:

$$\tau|_{U_i} = (-1)^i, \qquad k|_{U_i} = 2i - d + s$$

$$x U_i \subseteq U_{i+1}, \qquad y U_i \subseteq U_{i-1}$$

$$(U_{-1} = 0, \quad U_{d+1} = 0)$$

<u>Question</u> $q^2 = -1$

$$\mathcal{T} \hookrightarrow ?? \quad \text{embedding}$$

Hopf algebra

$$\downarrow$$

$$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$$

Drinfel'd polynomial $P_V(\lambda)$

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon')}$ aug. TD-alg.

$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$ tensor product of evaluation modules

$= \bigoplus_{i=0}^{d} U_i$, $d = l_1 + \cdots + l_n$ diameter, s: type

$U_i = \langle u_{j_1 \cdots j_n} = v_{j_1}^{(1)} \otimes \cdots \otimes v_{j_n}^{(n)} \mid i = j_1 + \cdots + j_n \rangle$

(i) $x U_i \subseteq U_{i+1}$, $y U_i \subseteq U_{i-1}$ $(U_{-1} = U_{d+1} = 0)$

(ii) $\dim U_0 = 1$

$y^i x^i \big|_{U_i} = \sigma_i \in \mathbb{C}$, $i = 0, 1, 2, \cdots$

$\sigma_0 = 1$, $\sigma_{d+1} = \sigma_{d+2} = \cdots = 0$

$\{\sigma_i\}_{i=0}^{d} \longleftrightarrow P_V(\lambda)$ monic polynomial of degree d

Drinfel'd polynomials

$\updownarrow$

the zeros of $P_V(\lambda)$

type I $\quad q^2 \neq \pm 1$

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(q-q^{-1})^{2i}([i]!)^2} \prod_{j=i+1}^{d} \left(\lambda - \varepsilon s^{-2} q^{2(d-j)} - \varepsilon^* s^2 q^{-2(d-j)}\right)$$

type II $\quad q^2 = 1$

case $(\varepsilon, \varepsilon^*) = (1, 1)$

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(i!)^2} \prod_{j=i+1}^{d} \left(\lambda - (j+s-d)^2\right)$$

case $(\varepsilon, \varepsilon^*) = (1, 0)$

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(i!)^2} \prod_{j=i+1}^{d} \left(\lambda - (j+s-d)\right)$$

case $(\varepsilon, \varepsilon^*) = (0, 0)$

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(i!)^2} \lambda^{d-i}$$

<u>type III</u> $\quad q^2 = -1$

$$P_V^{(0)}(\lambda) = \sum_{i=0}^{\left[\frac{d+1}{2}\right]} (-1)^i \frac{\sigma_{2i} + 16 i^2 \sigma_{2i-1}}{4^{3i} (i!)^2} \prod_{j=i+1}^{\left[\frac{d+1}{2}\right]} \left(\lambda - (2j-1+s-d)^2\right)$$

$$P_V^{(1)}(\lambda) = \sum_{i=0}^{\left[\frac{d}{2}\right]} (-1)^i \frac{\sigma_{2i}}{4^{3i} (i!)^2} \prod_{j=i+1}^{\left[\frac{d}{2}\right]} \left(\lambda - (2j+s-d)^2\right)$$

$$\underset{\deg d}{P_V(\lambda)} = P_V^{(0)}(\lambda) \cdot P_V^{(1)}(\lambda)$$

$\dashrightarrow P_V^{(0)}(\lambda) \quad \deg \left[\frac{d+1}{2}\right]$

$\dashrightarrow P_V^{(0)}(\lambda) \quad \deg \left[\frac{d}{2}\right]$

<u>Remark</u>

<u>type I</u> $\quad q^2 \neq \pm 1$

case $\quad (\varepsilon, \varepsilon^*) = (0,0)$

$\mathcal{T} \simeq$ the Borel subalgebra of $U_q(L(sl_2))$

$$\begin{array}{ccc} \mathcal{T} & \xhookrightarrow{\varphi_s} & U_q(L(sl_2)) \\ & & \swarrow \\ & V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n) & \end{array}$$

$\lambda^d P_V(\lambda^{-1})$ is the original Drinfeld polynomial up to scalar multiple.

Proposition

type I if $a \neq 0$

$$P_{V(\ell,a)}(\lambda) = \prod_{i=0}^{\ell-1} \left(\lambda + a q^{2i-\ell+1} + \varepsilon^* a^{-1} q^{-2i+\ell-1}\right)$$

if $a = 0$ in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$

$$P_{V(\ell,a)}(\lambda) = \lambda^\ell$$

type II

case $(\varepsilon, \varepsilon^*) = (1, 1)$

$$P_{V(\ell,a)}(\lambda) = \prod_{i=0}^{\ell-1} \left(\lambda - \left(a + \frac{2i-\ell+1}{2}\right)^2\right)$$

case $(\varepsilon, \varepsilon^*) = (1, 0)$

$$P_{V(\ell,a)}(\lambda) = \prod_{i=0}^{\ell-1} \left(\lambda + a + \frac{2i-\ell+1}{2}\right)$$

case $(\varepsilon, \varepsilon^*) = (0, 0)$

$$P_{V(\ell,a)}(\lambda) = (\lambda + a)^\ell$$

type III

$$P^{(0)}_{V(\ell,a)}(\lambda) = \prod_{\substack{0 \leq i \leq \ell-1 \\ i \equiv \ell-1 \bmod 2}} \left(\lambda - \left(a + \frac{2i-\ell+1}{2}\right)^2\right)$$

$$P^{(1)}_{V(\ell,a)}(\lambda) = \prod_{\substack{0 \leq i \leq \ell-1 \\ i \equiv \ell \bmod 2}} \left(\lambda - \left(a + \frac{2i-\ell+1}{2}\right)^2\right).$$

Theorem

$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$ tensor product of ev. modules for the aug. TD-alg. $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$

type I , type II

$$P_V(\lambda) = \prod_{i=1}^{n} P_{V(l_i, a_i)}(\lambda)$$

type III

$$P_V^{(0)}(\lambda) = \prod_{i=1}^{n} P_{V(l_i, a_i)}^{(\eta_i)}(\lambda),$$

$$P_V^{(1)}(\lambda) = \prod_{i=1}^{n} P_{V(l_i, a_i)}^{(1-\eta_i)}(\lambda)$$

where $\eta_i \equiv \sum_{\nu=i+1}^{n} l_\nu \mod 2 \quad (\eta_i \in \{0, 1\})$

$\eta_n = 0$.

finite dimensional irreducible $\mathcal{T}$-module

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ aug. TD-alg.

$V(\ell, a)$ evaluation module for $\mathcal{T}$

type I q-string of length ℓ

$$S(\ell, a) = \{ a q^{2i-\ell+1} \mid 0 \le i \le \ell-1 \}$$
$$= \{ a q^{-\ell+1}, a q^{-\ell+3}, \cdots, a q^{\ell-1} \}$$

case $(\varepsilon, \varepsilon^*) = (1, 0)$ and $a = 0$

$S(\ell, a)$: multi-set of ℓ zeros

$S(\ell, a)$, $S(\ell', a')$ in general position if

either (i) $S(\ell, a) \subseteq S(\ell', a)$

or $S(\ell', a) \subseteq S(\ell, a)$

or (ii) $S(\ell, a) \cup S(\ell', a)$ is not a q-string.

case $(\varepsilon, \varepsilon^*) = (1, 0)$

$S(\ell, 0)$, $S(\ell', a')$: in general position if $a' \ne 0$

not in general position if $a' = 0$

$S(\ell, a)$, $S(\ell', a')$ not in general position

$\Longleftrightarrow$

$S(\ell, a) \cup S(\ell', a')$ is a q-string of length $> \max\{\ell, \ell'\}$.

<u>type II</u> <u>string</u> of length l

$$S(l,a) = \left\{ a + \frac{2i-l+1}{2} \;\middle|\; 0 \leq i \leq l-1 \right\}$$
$$= \left\{ a - \frac{l-1}{2},\ a - \frac{l-3}{2},\ \cdots,\ a + \frac{l-1}{2} \right\}$$

<u>Rm</u>

$$S(l,-a) = \left\{ -a - \frac{l-1}{2},\ -a - \frac{l-3}{2},\ \cdots,\ -a + \frac{l-1}{2} \right)$$
$$\|$$
$$-S(l,a) = \left\{ -a + \frac{l-1}{2},\ -a + \frac{l-3}{2},\ \cdots,\ -a - \frac{l-1}{2} \right\}$$

$S(l,a)$, $S(l',a')$ <u>in general position</u> if

either (i) $S(l,a) \subseteq S(l',a')$

or $S(l',a') \subseteq S(l,a)$

or (ii) $S(l,a) \cup S(l',a')$ is not a string.

<u>type III</u> <u>signed string</u> of length l

$\eta \in \{0, 1\}$

$$S^{(\eta)}(l,a) = \left\{ \left(a + \frac{2i-l+1}{2},\ (-1)^{\eta+l-1-i}\right) \,\middle|\, 0 \le i \le l-1 \right\}$$

i.e.,
$$S^{(0)}(l,a) = \left\{ \left(a - \tfrac{l-1}{2}, (-1)^{l-1}\right), \cdots, \left(a + \tfrac{l-3}{2}, -1\right), \left(a + \tfrac{l-1}{2}, 1\right) \right\}$$

$$S^{(1)}(l,a) = \left\{ \left(a - \tfrac{l-1}{2}, (-1)^{l}\right), \cdots, \left(a + \tfrac{l-3}{2}, 1\right), \left(a + \tfrac{l-1}{2}, -1\right) \right\}$$

the <u>transpose</u> of $S^{(\eta)}(l,a)$

$$ {}^tS^{(\eta)}(l,a) = \left\{ \left(-a - \frac{2i-l+1}{2},\ (-1)^{\eta+l-1-i}\right) \,\middle|\, 0 \le i \le l-1 \right\}$$

$$= \begin{cases} S^{(\eta)}(l,-a) & \text{if } l \text{ is odd,} \\ S^{(1-\eta)}(l,-a) & \text{if } l \text{ is even} \end{cases}$$

$S^{(\eta)}(l,a)$, $S^{(\eta')}(l',a')$ <u>in general position</u> if

either (i) $S^{(\eta)}(l,a) \le S^{(\eta')}(l',a')$

or $S^{(\eta')}(l',a') \le S^{(\eta)}(l,a)$

or (ii) $S^{(\eta)}(l,a) \cup S^{(\eta')}(l',a')$ is not a signed string.

Main Theorem 1

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon')}$ aug. TD-alg.

$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$ tensor product of ev. modules

type I V : irreducible as a $\mathcal{T}$-module

$\Leftrightarrow$ (a) $P_V(\lambda) \neq 0$ for $\lambda = \varepsilon s^{-2} + \varepsilon^* s^2$, i.e., $\sigma_d \neq 0$

and

(b) case $(\varepsilon, \varepsilon^*) = (1,1)$

$S(l_i, a_i^{\varepsilon_i})$, $S(l_j, a_j^{\varepsilon_j})$ in general position

for all $i, j \in \{1, \cdots, n\}$, $i \neq j$

and for any $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$

case $(\varepsilon, \varepsilon^*) = (1, 0), (0, 0)$

$S(l_i, a_i)$, $S(l_j, a_j)$ in general position

for all $i, j \in \{1, \cdots, n\}$, $i \neq j$

Main Thm cont'

<u>type II</u> V : irreducible as a $\mathcal{T}$-module

$\Longleftrightarrow$

(a) $P_V(\lambda) \neq 0$ for $\lambda = \varepsilon\varepsilon^* s^2 + \varepsilon(1-\varepsilon^*)s$, i.e., $\sigma_d \neq 0$

and

(b) <u>case</u> $(\varepsilon, \varepsilon^*) = (1,1)$

$S(l_i, \varepsilon_i a_i)$, $S(l_j, \varepsilon_j a_j)$ in general position

for all $i, j \in \{1, \cdots, n\}$, $i \neq j$

and for any $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$

<u>case</u> $(\varepsilon, \varepsilon^*) = (1,0)$

$S(l_i, a_i)$, $S(l_j, a_j)$ in general position

for all $i, j \in \{1, \cdots, n\}$, $i \neq j$

<u>case</u> $(\varepsilon, \varepsilon^*) = (0,0)$

$a_i \neq a_j$ for all $i, j \in \{1, \cdots, n\}$, $i \neq j$.

Main Thm cont'

<u>type III</u> V : irreducible as a $\mathcal{T}$-module

$\Longleftrightarrow$

(a) $P_V^{(\eta)}(\lambda) \neq 0$ for $\eta \equiv d+1 \bmod 2$, $\lambda = s^2$, i.e., $\sigma_d \neq 0$.

and

(b) for all $i, j \in \{1, \cdots, n\}$, $i \neq j$

$S^{(\eta_i)}(l_i, a_i)$, $S^{(\eta_j)}(l_j, a_j)$: in general position

and

$S^{(\eta_i)}(l_i, a_i)$, ${}^tS^{(\eta_j)}(l_j, a_j)$: in general position

where

$$\eta_i \equiv \sum_{\nu=i+1}^{n} l_\nu \mod 2$$

$$\eta_j \equiv \sum_{\nu=j+1}^{n} l_\nu \mod 2.$$

Main Theorem 2

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ aug. TD-alg.

$V = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$ Tensor product of ev. modules

type s

$V' = V(l_1', a_1') \otimes \cdots \otimes V(l_{n'}', a_{n'}')$ tensor product of ev. modules

type s'

Assume V, V' are irreducible as $\mathcal{T}$-modules.

Then

$V \simeq V'$ as $\mathcal{T}$-modules

$\Longleftrightarrow$

(a) $s = s'$

and

(b) case type I, II

$P_V(\lambda) = P_{V'}(\lambda)$

case type III

$P_V^{(\eta)}(\lambda) = P_{V'}^{(\eta)}(\lambda)$ for $\eta = 0, 1$.

Main Theorem 3

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$: aug. TD-alg.

V : finite dimensional
irreducible $\mathcal{T}$-module

Then

$\exists\ V' = V(l_1, a_1) \otimes \cdots \otimes V(l_n, a_n)$ tensor product of ev. modules

$V \simeq V'$ as $\mathcal{T}$-modules.

$\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$: TD-alg.

$\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$: aug. TD-alg.

Given

$\iota_t : \mathcal{A} \longrightarrow \mathcal{T}$ embedding

$\rho_1 : \mathcal{T} \longrightarrow \mathrm{End}(V)$ f. d. irreducible representation

V : irreducible $\mathcal{T}$-module via ρ_1

type s, diameter d

determine when

$\rho_0 = \rho_1 \circ \iota_t : \mathcal{A} \longrightarrow \mathrm{End}(V)$ irreducible

and

the isomorphism class of the $\mathcal{A}$-module V via ρ_0.

<u>Eigenvalues</u>

$\mathcal{A} = \langle Z, Z^* \rangle$

$\{\theta_i\}_{i=0}^d$: eigenvalues of $\rho_0(Z)$ on V

$\{\theta_i^*\}_{i=0}^d$: eigenvalues of $\rho_0(Z^*)$ on V

type I

$$\theta_i = st\,q^{2i-d} + \varepsilon s^{-1}t^{-1}q^{-2i+d}, \qquad 0 \le i \le d$$

$$\theta_i^* = \varepsilon^* s t^{-1} q^{2i-d} + s^{-1} t\, q^{-2i+d}, \qquad 0 \le i \le d$$

type II

$$\theta_i = \frac{s+t+2i-d-1}{2}\left(\varepsilon\,\frac{s+t+2i-d-1}{2} + 1\right), \qquad 0 \le i \le d$$

$$\theta_i^* = \frac{s-t+2i-d-1}{2}\left(\varepsilon^*\,\frac{s-t+2i-d-1}{2} + 1\right), \qquad 0 \le i \le d$$

type III

$$\theta_i = (-1)^i (s+t+2i-d), \qquad 0 \le i \le d$$

$$\theta_i^* = (-1)^i (s-t+2i-d), \qquad 0 \le i \le d$$

Main Theorem 4

(1) V : irreducible as an A-module via $\rho_0 = \rho_1 \circ \iota_t$

$\Longleftrightarrow$

$$P_V(\lambda) \neq 0 \quad \text{for } \lambda = \begin{cases} t^2 + \varepsilon\varepsilon^* t^{-2} & (\text{type I}) \\ \varepsilon\varepsilon^* t^2 - \varepsilon(1-\varepsilon^*)t - (1-\varepsilon)(1-\varepsilon^*) & (\text{type II}) \\ t^2 & (\text{type III}) \end{cases}$$

(2) The isomorphism class of the A-modul V via $\rho_0 = \rho_1 \circ \iota_t$ is determined by

$$\left(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, P_V(\lambda)\right) \quad \text{for type I, II}$$

$$\left(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, (P_V^{(0)}(\lambda), P_V^{(1)}(\lambda))\right) \quad \text{for type III}.$$

Remark All f.d. irred. A-modules are obtained via $\rho_0 = \rho_1 \circ \iota_t$ for some ρ_1, ι_t.