On some classes of Deza graphs

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Definition

We consider the following generalization of strongly regular graphs.

Definition

Let v, k, b and a be integers such that $0 \leq a \leq b \leq k < v$. A graph Γ is a *Deza graph* with parameters (v, k, b, a) if

- Γ has exactly v vertices;
- for any vertex u in Γ its neighbourhood $\Gamma(u)$ has exactly k vertices;
- for any two different vertices u, w in Γ the intersection $\Gamma(u) \cap \Gamma(w)$ takes on one of two values b and a.

The only difference between a strongly regular graph and a Deza graph is that the size of $\Gamma(u) \cap \Gamma(w)$ does not depend on adjacency u and w.

Some history

These graphs were introduced by Antoine and Michel Deza.

A. Deza and M. Deza
The ridge graph of the metric polytope and some relatives
Polytopes: Abstract, convex and computational
T. Bisztriczky et al. (Editors). NATO ASI Series, Kluwer Academic. 1994, P. 359-372.

In the case $a = 0$ a Deza graph can have the diameter greater than 2, then this case Deza graph is considered separately. A *strictly Deza graph* **SDG** is a Deza graph which is not strongly regular **SRG** and has diameter 2.

Adjacency matrices

Let M be the adjacency matrix a graph Γ. Then Γ is a Deza graph with parameters (v, k, b, a) if and only if

$$M^2 = aA + bB + kI$$

for some $(0, 1)$-matrices A and B such that $A + B + I = J$, the all ones matrix. Note that Γ is a strongly regular graph if and only if A or B is M. As usual we used parameters (v, k, λ, μ) for a strongly regular graph. So we have the matrix equation

$$M^2 = \lambda M + \mu(J - M - I) + kI.$$

Some history

The study of strongly regular graphs has a long history, and the study of strictly Deza graphs started relatively recently. Significant results for strictly Deza graphs were obtained in the article written by five authors.

M. Erickson, S. Fernando, W.H. Haemers, W.H. Hardy, J. Hemmeter,
Deza graphs: A generalization of strongly regular graph
J. Combin. Designs 1999, V. 7, P. 395-405

Introduction

V.V. Kabanov invited his postgraduate student Galina Ermakova to investigate a class Deza graphs without triangles and a class Deza graphs without 3-cocliques.

As the complement of strongly regular graph is also strongly regular graph, these questions for strongly regular graphs are the same. But it is not true for Deza graphs.

There are exactly seven triangle-free strongly regular graphs known: the five cycle, the Petersen Graph, the Clebsch Graph, the Hoffman-Singleton Graph, the Gewirtz Graph, the Higman-Sims Graph, and a $(77, 16, 0, 4)$ strongly regular subgraph of the Higman-Sims graph. Every Moore Graph of diameter 2 is a triangle-free strongly regular graph, so if there is a 57-regular Moore Graph of diameter 2, this would add another to the list.

Moore graphs

A Moore graph is a regular graph of degree k and diameter d whose number of vertices equals to the upper bound

$$1 + k \sum_{i=0}^{d-1} (k-1)^i.$$

The Hoffman–Singleton theorem states that any Moore graph with girth 5 must have degree 2, 3, 7, or 57. The Moore graphs are:

The complete graphs K_n on $n > 2$ vertices. (diameter 1, girth 3, degree $n-1$, order n)

The odd cycles C_{2n+1}. (diameter n, girth $2n+1$, degree 2, order $2n+1$)

The Petersen graph. (diameter 2, girth 5, degree 3, order 10)

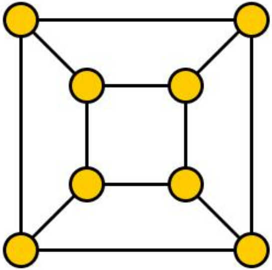
The Hoffman–Singleton graph. (diameter 2, girth 5, degree 7, order 50)

A hypothetical graph of diameter 2, girth 5, degree 57 and order 3250; it is currently unknown whether such a graph exists.

Unlike all other Moore graphs, Higman proved that the unknown Moore graph cannot be vertex-transitive. Machaj and Shiran and further proved that the order of the automorphism group of such a graph is at most 375.

Example

It is easy to see n-cube graph for $n > 2$ has the diameter n, girth 4, degree n and order 2^n. So it is a Deza graph with parameters $(2^n, n, 0, 2)$.
These Deza graphs do not have triangles. Note, that the complement of n-cube is a strictly Deza graph without 3-cocliques with parameters $(2^n, 2^n - n - 1, 2^n - 2n, 2^n - 2n - 2)$.
It's clear Deza graphs without 3-cocliques are coedge regular graphs. So if Γ such Deza graph with parameters (v, k, b, a), then $\mu(\Gamma) \in \{a, b\}$.



Graphs without 3-cocliques

Ermakova proved if Deza graph with parameters (v, k, b, a) without 3-cocliques that in case $\mu(\Gamma) = b$ we have $b \in \{a+1, a+2\}$. In the case $b = a+2$ the complement of Γ is an amply regular graph with parameters $(v, v-k-1, 0, 2)$.

Let $v-k-1 = l$. It is interesting amply regular graphs with parameters $(v, l, 0, 2)$ were investigated before by Andries E. Brouwer for degree l less than 8.

Andries E. Brouwer. Classification of small $(0, 2)$-graphs Journal of Combinatorial Theory, Series A 113 (2006) 1636–1645 www.elsevier.com/locate/jcta

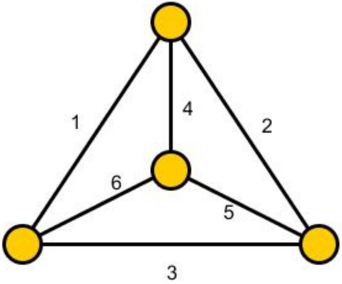
Andries E. Brouwer, P. R. J. Ostergard find the 302 graphs of degree 8.

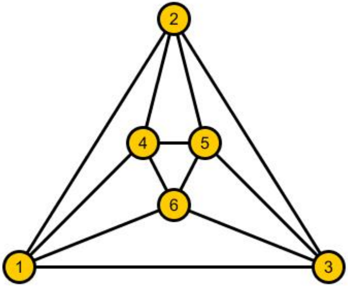
It is also known amply regular graph with parameters $(v, l, 0, 2)$ whose diameter equals to valency is n-cube.

On this conference we have abstract of Ahkhamova about Deza graphs without 3-coclique with $\mu(\Gamma) = a$ where $1 \le a \le 3$.

Definition

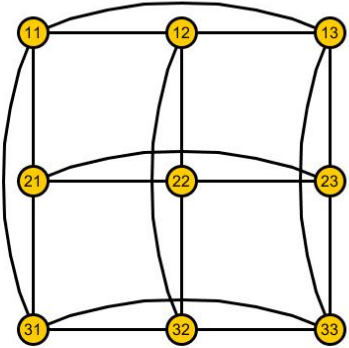
For a given graph Γ, its *line graph* $L(\Gamma)$ is the graph which vertices are edges of the graph Γ, and two vertices are adjacent if and only if the corresponding edges have exactly one common vertex in Γ.





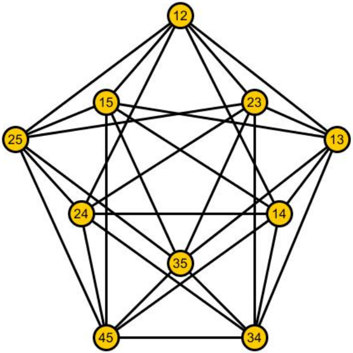
Lattice graph

For a positive integer n, *the lattice graph* $L(n)$ is the graph with vertex set $\{1,\dots n\}^2$ in which vertex (a,b) is connected to vertex (c,d) if $a=c$ or $b=d$. Thus, the vertices may be arranged at the points in an $n\times n$-grid, with vertices being connected if they lie in the same row or column. Alternatively, we can understand this graph as the line graph of a bipartite complete graph between two sets of n vertices. It is routine to see that the parameters of this graph are: $v=n^2$, $k=2(n-1)$, $\lambda=n-2$, $\mu=2$.



Triangular graph

For a positive integer n, *the triangular graph* $T(n)$ may be defined to be the line graph of the complete graph on n vertices. In over words, its vertices are the subsets of size 2 of $\{1 \ldots n\}$. Two of these sets are connected by an edge if their intersection has size 1. It is routine to see that the parameters of this graph are: $v = \frac{n(n-1)}{2}$, $k = 2(n-2)$, $\lambda = (n-2)$, $\mu = 4$.

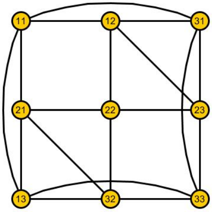


It is a well known fact in the theory of strongly regular graphs, that no other strongly regular line graph does exist.

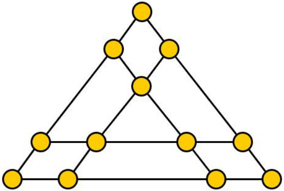
This result generalized to Deza graphs.

Line strictly Deza graphs

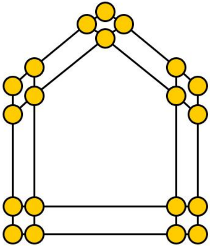
For convenience of the formulation, introduce the following notation.
Denote by Δ_1 the Deza graph with parameters $(9, 4, 2, 1)$ presented in the next picture (a).
Denote by Δ_2 the Deza graph with parameters $(12, 6, 3, 2)$ presented in the next picture (b).
Denote by Δ_3 the Deza graph with parameters $(20, 6, 2, 1)$ presented in the next picture (c).



(a)



(b)



(c)

Line sdg graphs

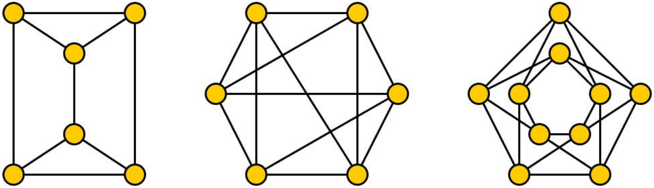
The following theorem gives us the complete classification of strictly Deza line graphs.

Theorem (V.V. Kabanov, A. Mityanina, 2010)

A graph Γ is strictly Deza line graph if and only if it is

1. *the $4 \times n$-lattice, where $n > 1$, $n \neq 4$;*
2. *one of the graphs Δ_1, Δ_2, or Δ_3.*

Note that the $4 \times n$-lattice is the line graph for the complete bipartite graph $K_{4,n}$. For $n = 4$, this graph is strongly regular. The graphs Δ_1, Δ_2, and Δ_3 are the line graphs of the graphs presented in the (a), (b), and (c), respectively.



(a) (b) (c)

Construction from strongly regular graph

Theorem (M. Ericson, S. Fernando, W.H. Haemers, D. Hardy and J. Hemmiter, 1998)

Let Γ *be a* $(n, k, \lambda, \mu)-$*SRG with* $k \neq \mu$, $\lambda \neq \mu$ *and adjacency matrix* M. *Let* P *be a permutation matrix. Then* PM *is the adjacency matrix of the Deza graph* Γ' *if and only if* $P = I$ *or* P *represents an involution of* Γ *(i.e. an automorphism of order two) that interchanges only nonadjacent vertices. Moreover,* Γ' *is strictly Deza graph if and only if* $P \neq I$, $\lambda \neq 0$ *and* $\mu \neq 0$.

Lattice and triangular graphs

It it well known that lattice graphs and triangular graphs determine by their parameters in srg class.

But for Deza graphs which obtained from srg it is not true. For example there is two graphs with parameters $(9, 4, 2, 1)$, and only one of them may be obtained from $L(3)$. That is why we characterized this graphs by their parameters and local structure.

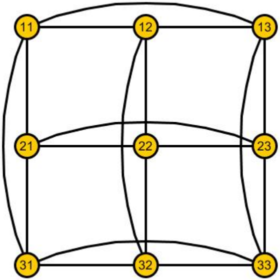
Involutions of $L(n)$

Theorem (V.V. Kabanov, L. S.)

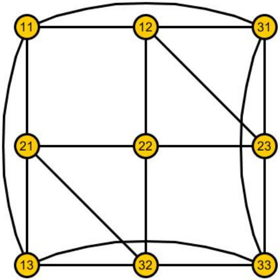
If n is even then there are two involutions of $L(n)$ as required (up to ordering of vertices). The first involution fixes n pairwise nonadjacent vertices and can be considered as the symmetry with respect to the main diagonal. The second involution doesn't have fixed vertices and can be considered as the superposition of symmetries with respect to main and secondary diagonals. If n is odd then $L(n)$ admits an involution of the first type only.

Denote the first involution by Φ_1 and the second involution by Φ_2, and denote the corresponding Deza graphs by $\Phi_1 L(n)$ and $\Phi_2 L(n)$.

Example



$L(3)$



$\Phi_1 L(3)$

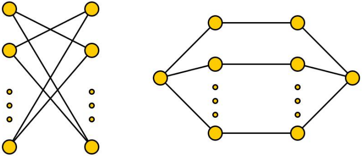
Local structure of $\Phi_1 L(n)$

Definition

Let F be a set of graphs, then Γ is a *locally-F graph* if and only if for every $x \in \Gamma$ $\Gamma(x) \in F$ and for every $H \in F$ there is $x \in \Gamma$: $\Gamma(x) \simeq H$.

Lemma (V.V. Kabanov, L. S.)

A graph $\Phi_1 L(n)$ is the locally F-$(n^2, 2(n-1), n-2, 2)$-DG, where $F = \{F_1, F_2\}$.



F_1 F_2

Characterization of $\Phi_1 L(n)$

Theorem (V.V. Kabanov, L. S.)

A locally F*-*$(n^2, 2(n-1), n-2, 2)$*-DG, where* F *from previous lemma, is isomorphic to* $\Phi_1 L(n)$*.*

Involutions of $T(n)$

Lemma (L. S.)

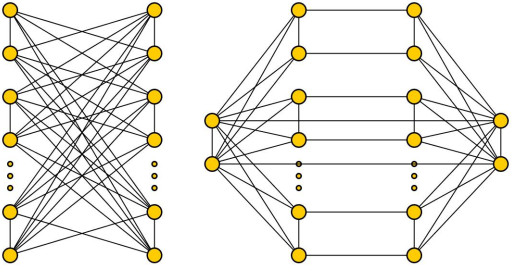
If n is even; then there is a unique involution of $T(n)$ as required (up to ordering of vertices). It fixes $n/2$ pairwise nonadjacent vertices and interchanges any pair of cliques that have a common fixed vertex. If n is odd there is no required involutions.

Denote this involution by Ψ and denote the corresponding Deza graph by $\Psi T(n)$.

Local structure of $\Psi T(n)$

Lemma (L. S.)

A graph $\Psi T(n)$ is locally F-$(\binom{n}{2}, 2(n-2), n-2, 4)$-DG, where $F = \{F_1, F_2\}$, in particular, the fixed vertices have a neighborhood isomorphic to F_1, the non-fixed vertices have neighborhood isomorphic to F_2.



F_1 F_2

Characterization of $\Psi T(n)$

Theorem (L. S.)

A locally F-$(\binom{n}{2}, 2(n-2), n-2, 4)$-*DG, where* F *from Lemma 2, is isomorphic to* $\Psi T(n)$.

Other results

In work of S. Goryainov and L. S. were characterized by their parameters and local structure deza graphs which obtained from complements to $T(n)$ and $L(n)$.

List of Deza graphs on at most 13 vertices

In work of M. Ericson, S. Fernando, W.H. Haemers, D. Hardy and J. Hemmiter was found list of Deza graphs with at most 13 vertices.

Parameters	Constractions
(8,4,2,0)	$K_4 \times K_2$
(8,4,2,1)	Cayley graph of group C_8
(8,5,4,2)	Cayley graph of group C_8
(9,4,2,1)	Cayley graph of group C_9.
(9,4,2,1)	with involution from $T(3)$
(10,5,4,2)	Cayley graph of group C_{10}
(12,5,2,1)	Cayley graph of group C_{12}
(12,6,3,2)	Cayley graph of group A_4
(12,6,3,2)	
(12,7,4,3)	Cayley graph of group C_{12}
(12,7,6,2)	Cayley graph of group C_{12}
(12,9,8,6)	Cayley graph of group C_{12}
(13,8,5,4)	Cayley graph of group C_{12}

List of Deza graphs on 14-16 vertices

In work of S. Goryainov and L. S. was continued this search and were found all deza graphs with 14-16 vertices.

Parameters	Constructions
(14,9,6,4)	Cayley graph of group D_{14}
(15,6,3,1)	with involution from $\overline{T(6)}$
(16,5,2,1)	Cayley graph of group QD_{16}
(16,7,4,2)	Cayley graph of group $C_4 \times C_4$
(16,7,4,2)	Cayley graph of group $C_4 \times C_4$
(16,8,4,2)	Cayley graph of group C_{16}
(16,9,6,4)	with involution from $\overline{L(4)}$
(16,9,6,4)	with involution from $\overline{L(4)}$
(16,9,8,2)	Cayley graph of group C_{16}
(16,11,8,6)	Cayley graph of group $C_4 \times C_4$
(16,12,10,8)	Cayley graph of group C_{16}
(16,13,12,10)	Cayley graph of group C_{16}

Definition

Cayley graph of group G with generating set S is graph which vertices are elements of G, and $x \sim y$ iff $xy^{-1} \in S$. Denote $Cay(G, S)$. If cayley graph undirected without loops then $id \notin S$ and $S^{-1} = \{g^{-1} | \ g \in S\} = S$.
If graph is Deza graph and Cayley graph then we call it *Cayley-Deza graph*.
For example $Cay(C_8, \{1, 2, 6, 7\})$ is Deza graph with parameters $(8, 4, 2, 1)$.

Results

In work of S. Goryainov and L. S. were found all Cayley-Deza graphs on at most 59 vertices. We obtained two lists, the first contains for each group Cayley-Deza graphs for this group, the second contains groups for each Cayley-Deza graph of which this graph can be obtained.

Strongly regular graphs

Strongly regular graphs with disconnected second neighborhood were classified in work Gardiner A.D., Godsil C.D., Hensel A.D., Royle G.F.

Theorem

Let Γ *be a strongly regular graph. For any* $u \in V(\Gamma)$, *if* $\Gamma_2(u)$ *is disconnected, then it contains no edges and* Γ *is a complete multipartite graph (with parts of the same size* $s > 2$*).*

The following question is naturally arises. What could be strictly Deza graphs with disconnected second neighborhood?

Vertex-transitive graphs

Theorem (S. Goryainov, L. S.)

A vertex-transitive Deza graph with disconnected second neighborhood is either edge-regular or coedge-regular.

Coedge-regular graphs

Theorem (S. Goryainov, G. Isakova)

Let Γ be a coedge-regular Deza graph of diameter 2. If there exists $u \in \Gamma$ such that $\Gamma_2(u)$ is disconnected then Γ is either a complete multipartite graph with parts of the same size $s > 2$ or its 2-clique-extension $\Gamma[K_2]$.

Edge-regular graphs

Theorem (N. Maslova)

Let Γ be an edge-regular Deza graph of diameter 2. If there exists $u \in \Gamma$ such that $\Gamma_2(u)$ is disconnected then Γ is either a complete multipartite graph with parts of the same size $s > 2$ or $\Gamma \cong \Delta_1[\Delta_2]$ where Δ_1 is a strongly regular graph with $\lambda = \mu$ and Δ_2 is a coclique of size $s \geq 2$.

Thank you!