Group factorizations, graphs and related topics

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1 Factorizations of groups

By famous Burnside's p^{α}-Lemma, a group G, having the conjugacy class of prime power size, is non-simple. This implies immediately the solubility of groups of order $p^a q^b$, where p and q are prime numbers. Clearly, in the case of Burnside's Lemma the group G has a factorization of the form $G = C_G(x)P;$ where $|G : C_G(x)|$ is a power of a prime p and P *in* $Syl_p(G):$

This result opens a wide road in the theory of finite groups, having links with many directions in it. First of all, if $|G| = p^a q^b$ is a product of a powers of different primes p and q, then $G = PQ$ is a product of its Sylow p-subgroup and q-subgroup.

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First of all this leads to a conjecture that the nite product $G = AB$ of two nilpotent subgroups A and B is soluble. This problem was solved by H. Wielandt in 1958 (in fact, he started in 1951) in the case when A and B are of coprime orders. In 1961 his student O.H. Kegel proved the general result removing the arithmetical condition $(|A|, |B|) = 1$ by a very elegant lemma.

The solubility of a finite group G = AB in the case, when A and B have nilpotent subgroups $A_0 \leq A$ and $B_0 \leq B$ having index at most 2 in the corresponding groups was proved by L.Kazarin (On the products of two groups that are closed to being nilpotent, Mat.Sbornik 110(1979), 51-65). An easy examples shows that one cannot further relax the conditions on A and B.

Note that this result was proved after many attacks by several authors (V.S. Monakhov, A.V. Romanovskii and others).

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In 1973 E. Pennington (On products of nite nilpotent groups. Math.Z. 134 (1973), 81-83) has proved that $G^{\alpha+\beta}$, the α+β -term of the derived series of *G* belongs to Fitting subgroup of *G*, provided *A* and *B* have nilpotency classes and respectively. F. Gross (Finite groups which are the products of two nilpotent groups, Bull. Australian Math.Soc. 9(1973), 267-274) proved that in this case even $G^{\alpha+\beta} \leq \Phi(G)$.

The natural conjecture that the derived length of a product *G* = *AB* of a nilpotent groups *A* and *B* of nilpotency classes and respectively is bonded by α+β was disproved by J. Cossey and S. Stonehewer (Bull. London Math. Soc. 30, N 144 (1998), 247-250).

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Note that in 1955 N. Ito (Űber das Product von zwei abelshen Gruppen, Math.Z. 62 (1955), 400 - 401) has proved that the product of two abelian groups has derived length at most two. The surprisingly short proof of Ito's theorem, which is valid for all groups without any further restrictions was recovered later by N.S. Chernikov (Products of almost abelian groups. Ukrain. Mat. Z.33 (1981), 136 - 138). He proved that a group $G = AB$ which is a product of two subgroups, having central subgroups $Z(A)$ and $Z(B)$ of finite indexes in A and B respectively, has a soluble normal subgroup S of finite index with derived length bounded by a function of $|A : Z(A)|$ and $|B : Z(B)|$. However the general question about the product of almost abelian subgroups A and B is still open. The partial positive results where obtained by Ya. Sysak (1988) and J.S. Wilson (1990).

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If a finite group $G = AB$ is a product of an abelian group A and a group B, then $Z(B)$ is contained in a soluble normal subgroup of G (L. Kazarin, On a problem of Szep, Mathematics in the USSR-Izvestia, 28:3(1987), 486 - 495). If the orders of A and B are coprime, the result does not depend on the classication of nite simpke groups (CFSG). Moreover the normal closure of $Z(B)$ has derived length at most two. In the case of a Burnside's Lemma the normal closure of an element, having a power of a prime p conjugates, (L. Kazarin, Burnsides p^{α} - lemma, Math. Notes 48:2(1990), 749 - 751) has a soluble normal closure in G. Later A. and R. Camina's (Implications of conjugacy class size. J. Group Theory, 1 (1998), 257 - 269) have proved that the commutator subgroup of that element is a p-group.

1 Factorizations of groups

When the classification of finite simple groups was announced, many results were obtained using CFSG. One of the remarkable results was the description of factorizations of all finite almost simple groups by maximal subgroups due to M. Liebeck, Sh. Praeger and J. Saxl (Mem.Amer. Math. Soc. 432 (1990), iv, 1-151). The memoir contains also a lot of useful information on finite simple groups in a very clear form. Slightly before L. Kazarin classied (using CFSG) all possible composition factors of a group $G = AB$, which is a product of two soluble subgroups A and B. It is interesting that the main idea was exactly the same as in the memoirs of the previous authors.

Next result by B. Amberg and L. Kazarin (On the product of a nilpotent group and a group with non-trivial center. J. Algebra, 315 (2007), 69 - 95) is a generalization of a classical theorems by Burnside, Kegel and Wieland cited above.

Theorem 1.1 *Let a finite group $G = AB$ be a product of a nilpotent group A and a group B. Then the normal closure of a subgroup $Z(B)$ in G is soluble.*

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Another generalizations were obtained in a series of works by L. Kazarin, A. Martinez-Pastor and M.D. Perez-Ramos. This started in 2007 and the final part is published this year (On the product of two n-decomposable groups, Revista Matematica Iberoamericana 31(2015), 33 - 50).

Recall that the group X is called a π -decomposable for a certain set π of primes (π′ is a complement of π in the set of all primes), provided $X = O_{\pi}(X) \times O_{\pi'}(X)$. For instance, every finite nilpotent group is π - decomposable for each set of primes.

Theorem 1.2 *Let G = AB be a finite product of two π–decomposable groups A and B. If π contains only odd primes, then* $O_{\pi}(A)O_{\pi}(B)$ *is a Hall π -subgroup of G.*

The preliminary version of this theorem states that the finite group $G = AB$, which is a product of a π -decomposable group A and a π′-group B has a normal subgroup $O_{\pi}(A)$, provided π consists of odd primes only.

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Recently E.M. Palchik (On finite factorized groups, Trudy Inst. Matematiki i Mekhaniki UrO RAN.19:3 (2013),261-267) has proved the following generalization of this result.

Theorem 1.3 *Let G = AB be a finite simple group which is a product of a π-soluble group A and a π-group B. If π contains only odd primes, and A is a π-soluble non-soluble group, then (|A|; |B|) = 1 and G is the group listed in theorem due Z. Arad and E. Fisman (J.Algebra 96(1984),522 - 548).*

Note that this will be wrong, if one omit the simplicity condition on *G*.

C.H. Li and B. Xia obtain a generalization of Kazarin's result on the products of soluble groups. In arXiv:1408.0350v1[math.Gr]2Aug 2014 they classied almost simple groups *G* = *AB* with soluble factor *A*. In particular, this generalized Theorem 1.3.

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Next is a generalization of both Theorems 1.2 and 1.3. by L. Kazarin, A.Martinez-Pastor and M.D. Perez-Ramos:

Theorem 1.4 *Let $G = AB$ be a nite product of two π-soluble group A and a π -group B. If π contains only odd primes, then the nonabelian composition factors of G are either in the list of Z. Arad and E. Fisman (J.Algebra 96 (1984), 522 - 548), or in the list of L. Kazarin's theorem (On groups which are the products of two solvable subgroups. Commun. Algebra, 14 (1986) 1001 -1066).*

H. Wielandt (J.Austral. Math.Soc. 1(1960), 143 - 146) proved that a finite group G, which is a product of soluble subgroups $A;B;C$ of pairwise coprime indices is soluble. In this case $G = AB = AC = BC$. P. Hall (1928) has found such factorization in any soluble group with order divisible by 3 different primes. L. Kazarin improved Hall's theorem in "Factorization of finite group by solvable subgroups\, Ukrain, Math.J. 43(1992), 883 -886, deleting the arithmetic condition in Wielandt's theorem.

O.H. Kegel (Math.Z. 87(1965) has proved that a finite group G having factorization $G = AB = AC = BC$ with nilpotent subgroups $A;B$ and C is nilpotent as well.

2 Graphs on the sets of primes

Let $x > 1$ be a natural number. Then $\pi(x)$ is the set of prime divisors of x. If X is a set of natural numbers, then $\rho(X) = U_{x \epsilon X} \pi(x)$. Denote the graph $\Gamma(X)$ with the set $\rho(X) = V(X)$ of vertices. Two vertices $p, q \epsilon V(X)$ are adjacent if $p, q \epsilon (X)$ and $pq | x$ for some $x \epsilon X$. Another graph $\Delta(X)$ on the set X denoted as follows. Vertices a and b in X are adjacent, if the greatest common divisor of a and b is bigger than 1.

The following argument due to M.L. Lewis shows the relation between two types of graphs.

Lemma 2.1 *Let X be a set of natural numbers a,b ϵ X. If p|a; q|b, then a and b are in the same component of the graph Δ(X) if and only if p and q belongs to the same component of the graph Γ(X). In this case the distances between a and b in these graphs differs at most 1. I.e.* $|d_{\Gamma(X)}(a; b) - d_{\Delta(X)}(p; q)| \leq 1$. *Moreover, if one of the graphs is connected, their diameters differs not more than by 1.*

2 Graphs on the sets of primes

It seems that first graphs in group theory after Cayley were graphs invented by S.A. Chounikhin ("On the existence of subgroups in finite groups". Trudy seminara po teorii grupp, M.-L., Gostechizdat, 1938) and Gruenberg-Kegel graphs, became famous since the paper by J.S. Williams (Prime graph components of finite groups, J.Algebra 69(1981), 487 - 513). Chunikhin's graph was defined by L.Kazarin in 1978.

The main arithmetical parameters in finite group theory are of prime interest. They are as follows: the sizes of the conjugacy classes of elements ($cs(G)$, the degrees of irreducible characters of a group ($cd(G)$) and the set of orders of elements in a group ($\omega(G)$. The last parameters are also of interest for innite periodic groups (Burnside's groups).

Correspondingly, we have the following prime graphs with the set $X = (|G|)$:

2 Graphs on the sets of primes

1.Chounikhin graph $\Gamma_{ch}(G)$ is defined on the set cs(G) of sizes of conjugacy classes of elements of *G*. This is a graph of type Δ in a terminology above. Originally, the vertices$|K1|$ and $|K2|$ are defined as adjacent, if $(|K1|; |K2|) = 1$, i.e. studied the dual graph $D(\Gamma_{ch}(G))$. In these therms S.A. Chounikhin has proved, that a finite group whose graph $D(\Gamma_{ch}(G))$ has a triangle, is not simple. This result and Burnside's p^{α}-lemma leads him to a conjecture that the group having two conjugacy classes with relatively prime sizes is not simple.

Followig Chunikhin, in 1980 L. Kazarin has found the structure of a group, having two isolated classes of a group *G*. This means that the group *G* has two classes *K* and *K′* of relatively prime sizes, dierent from 1, such that the size of any other class is coprime with *K* or with *K′*. Later this result was repeated by E.A. Bertram, M.Herzog and A.Mann (1990).

In the paper "On S.A.Chunikhin problem" (Investigations in theory of groups, Sverdlovsk 1984, p.81-99) L. Kazarin proved that a non-abelian finite simple group has a complete graph Γ_{ch}. More general result was obtained later by Z. Arad and E.Fisman. They have proved in J.Algebra 108(1987), 340 -354 that the finite simple non abelian group *G* has no factorization $G = AB$ with $Z(A) \neq 1 \neq Z(B)$.

The investigations on restrictions on the graph Γ_{ch} are popular now. See recent papers by A. Beltran, M. Felipe, A.and R. Camina's. In this area there is a question by J. Thompson:

Is it true that the set $\{cs(g) | g \in G\}$ determines every simple group up to isomorphism?

Some results in this direction were obtained by Chen, A.V. Vasiliev and I.B.Gorshkov.

2 Graphs on the sets of primes

2. Gruenberg-Kegel prime graph and related topics

This graph is defined on the set $\rho(X)$ of prime divisors of orders of elements in G. Two vertices p and q are adjacent in $\rho(X) = U_{x\epsilon X}\pi(x)$ =π(G) if there exists an element x *in G* of order pq. In his paper (Prime graph components of nite simple groups, Mathematics of the ISSR-Sbornik 67:1(1990), 235 -247) A.S.Kondratiev determined connected components of all simple groups of Lie type of even characteristic. This was finishing the previous work by J. Williams. This information became very useful in the description of composition factors of finite groups, which are the product of two its soluble subgroups by L. Kazarin (On groups which are the products of two solvable subgroups. Communications in Algebra, 14 (1986),1001 -1066).

Later on this graph became important in the recognition finite groups by its spectrum's $\omega(G)$. Very useful was the fact that certain subsets of vertices form cliques in the dual graph of size at least 3 (independent vertices) in certain simple groups. The maximal independent subsets of vertices in Gruenberg-Kegel graphs of all simple groups were found by A.V. Vasiliev and E.P. Vdovin.

In recent paper in Vladikavkazskii matematicheskii zhournal (17:2(2015), 47-55) V.D. Mazurov listed finite simple groups G that are known as not recognizable by spectrum. They are recognizable by spectrum and order. If we consider spectrum of the group together with its graph Γ*sol* introduced by S. Abe and N. Iiyori (see next subsection), then the following groups became recognizable:

$U_3(3); U_3(7); U_4(2); U_5(2); M_{11}; M_{22}; L_3(3); A_{10}; S_4(2)$.

2 Graphs on the sets of primes

3. Graph $\Gamma_{sol}(G)$ invented by S. Abe and N. Iiyori (A generalization of prime graphs of nite groups), has the same number of vertices (G), but primes p and q are adjacent if there exists in G a soluble subgroup, whose order is divisible by pq. Abe and Iiyori has proved that the graph $\Gamma_{sol}(G)$ is always connected and they described all finite groups G in which $\Gamma_{sol}(G)$ has a clique on all odd primes. B. Amberg, A. Carocca and L. Kazarin in "Criteria for the solubility and non-simplicity of nite groups", J. Algebra,285(2005), 58 - 72 described finite simple groups G such that the subgraph on the vertices $\pi(G) \setminus \{2,3\}$ form a clique. The following result from this paper is surprising:

Theorem 2.2 *Let G be a finite simple group and let p be the largest prime divisor of |G|. Then there exists a vertex* $q \in \pi(G) \setminus \{p\}$*, which is not adjacent with p in* $\Gamma_{sol}(G)$.

The largest number of pairwise independent vertices in Gruenberg-Kegel graph $\Gamma(G)$ denote by $t(G)$, and the largest pairwise independent number of vertices in $\Gamma_{sol}(G)$ denote by $t_s(G)$. Clearly, $1 \le t_s(G) \le t(G)$. By the result due Abe and Iiyori, for every simple group G one has $t_s(G) \ge 1$. B. Amberg and L. Kazarin determined bounds for the parameters $t_s(G)$ for all finite simple groups(see the paper "On the soluble graph of a finite simple group" in Communs. Algebra, 41(2013), 2297 - 2309). In particular, this gives the list of all simple groups G with $t_s(G) = 2$.

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Theorem 2.3 *Let G be a finite simple group such that the dual graph to* $\Gamma_{sol}(G)$ *has no triangles (i.e.* $t_s(G) = 2$*). Then G is isomorphic to one of the following groups:*

1. $L_n^{\epsilon}(q)$*, where either* $n \le 4$*, or* $n \le 7$*;* $q = 2$*, or* $5 \le n \le 6$*;* $q_4 = 5$*;*

2. $S_4(q)$*,* $P\Omega_8^{+}(2)$*,* $^3D4(2)$*,*$^2F_4(2)'$*,*$G_2(3)$*, or* $S_6(2)$*;*

3. M_{11}*,* M_{12}*,* M_{22}*, HS, McL,* J_2 *or* A_n*,* $n \le 10$*.*

Here q_4 is the largest prime divisor of $q^2 + 1$, $\epsilon = \pm$ is a sign, $L_n^{+}(q) = L_n(q)$ and $L_n^{-}(q) = U_n(q)$.

The proof uses two papers by A.V. Vasiliev and E.P. Vdovin (Algebra and Logic 44(2005), 381 - 406 and Algebra and Logic 50(2011), 265 - 284). Note that M. Xagie has proved that the diameter of the graph $\Gamma_{sol}(G)$ is at most 4 (this is true for M_{23}) and proved that every sporadic simple group is recognizable by its graph $\Gamma_{sol}(G)$ and its order.

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Note the following corollaries.

Theorem 2.4 Let G be a finite group in which every subgroup of odd prime order is c-permutable with respect to the family **S** of all Sylow subgroup of G. Then the soluble radical $S(G)$ is non-trivial.

Theorem 2.5 Let G be a nite simple group with a soluble subgroups A and B such that $\pi(G) = \pi(A) \cup \pi(B)$. Then G is one of the groups in the list of Theorem 2.3.

Theorem 2.5 gives the opportunity to obtain more simple proof of a theorem of Kazarin, which described the composition factors of a finite group which is a product of two solvable subgroups.

2 Graphs on the sets of primes

3. Graph $\Gamma_A(G)$ was studied by L. Kazarin, A. Martinez-Pastor and M.D. Perez-Ramos in the paper "On the Sylow graph of a group and Sylow normalizers ", Israel J.Math. 188 (2011), 251 271. This graph of a group G again has the set $\pi(G)$ as a set of vertices. Two vertices $p; q \in \pi(G)$ are adjacent if for Sylow p subgroup P of G and Sylow q-subgroup Q of G the order of a group $N_G(P)/PC_G(P)$ is divisible by q. One of the main results concerning these graphs is as follows:

Theorem 2.6 *Let G be a finite almost simple group. Then the graph* $\Gamma_A(G)$ *is connected*.

It terns out that the graph $\Gamma_A(G)$ is a subgraph of the graph $\Gamma_{sol}(G)$.Therefore the connectivity of $\Gamma_{sol}(G)$ in general is the corollary of Theorem 2.6. Indeed, if G is not simple, then $\Gamma_{sol}(G)$ is connected. But if G is simple, then the result follows from Theorem 2.6.

2 Graphs on the sets of primes

4. Graph $\Gamma_{Syl}(G)$. It is natural to define such type of graphs as follows: the set of vertices of $\Gamma_{Syl}(G)$ is the set $\pi(G)$. The vertices p and q in $\pi(G)$ are adjacent if G has a Hall $\{p, q\}$-subgroup.

It was proved by V. Tyutyanov and T. Tikhonenko the following result (2010):

Theorem 2.7 *Let G be a group. If the graph $\Gamma_{Sol}(G)$ has a vertex 3 adjacent with all other vertices, then the nonabelian composition factors of G belongs to the following list:*
$L_2(7); U_3(q)((q - 1); 9) = 3$ and $q = 0(mod 4)$; or $q = -1(mod 4)); Sz(q)$.

3 ABA-factorizations

We say that *G* posseses an ABA-factorization, if every element *g* in *G* can be wrritten in the form *g* = *aba′*, where *a; a′* belong to a subgroup *A* and *b* belongs to a subgroup *B*. Groups of this type started to study D.Gorenstein and I.N. Herstein (1959). They consider the case, when *A* and *B* are cyclic. If |*A*| and |*B*| are coprime it was proved that *G* is soluble.
M. Guterman (1969) considered the case of abelian subgroups *A* and *B*. Using Walter's classication of simple groups with abelian Sylow 2-subgroup, he proved that *G* = *ABA* is soluble, when |*A*| and |*B*| are coprime and *B* is of odd order.

Ya.P. Sysak proved the solubility of a group *G* = *ABA* for several cases(1982 - 1988). For instance, when *A* is abelian, *B* is cyclic primary groupand *A* and *B* are of coprime orders.

D. Zagorin and L. Kazarin (1996) announced solubility of a group *G* =*ABA* for abelian *A* and cyclic *B*. Moreover, D. Zagorin has proved in his thesis that simple groups possessing *ABA*-factorizations, are isomorphic to $L_2(q)$ for even *q*.

Independently this result was proved by E.P. Vdovin.
Recently Sh. Praeger and Y. Alavi published the program of the descrition of *ABA*-factorizations of finite groups.

3 ABA-factorizations

Note that every 2-transitive permutation group is an *ABA*-group, where *A* is a point stabilizer and *B* is a cyclic group, not contained in *A*.Another important examples are groups of Lie type having a factorizationsof the forn $G = BNB$, where *B* is a Borel subgroup and *N* is a subgroup of the normalizer of a Cartan subgroup *C*. In particular, $N/C=W$, is the Weyl group.

Examples 1. The alternating group $G = A_5 = V_4D_6V_4$ is an *ABA* product with an abelian *A* and metacyclic *B*.

2. The symmetric group $S_5 = D_8C_6D_8$ is an *ABA* product with nilpotent $A \simeq D_8$ and cyclic $B \simeq C_6$

3. (Ya. Sysak) The symmetric group $A_6 = ABA$ with *A* its Sylow 2-subgroup and *B* a dihedral group of order 8.

4. An alternating group $G = A_6 = ABA$, where *A* is its Sylow 3-subgroup and is a dihedral group of order 8.

A very important class of examples was discovered by D.G. Higman and J.E. McLaughlin.Let *G* be an automorphisms group of a flag-transitive geometry, where *A* and *B* are the stabilizers of a point and line respectively (flag is a pair of incident point and line). This group $G = ABA$ exactly, when every pair of point incident at least one line.

Sporadic simple groups with a non-trivial factorization of form $G = AB$ are as follows:

$M_{11};M_{12};M_{22};M_{23};M_{24}; J_2;HS;Ru;He; Suz; Fi_{23};Co_1$.

Recall that the group Co_3 is 2-transitive. Also the groups J_2, *McL*, *HS*, *Ru* and *Suz* have non-trivial *ABA* factorizations. Therefore at least 15 sporadic groups have *AB* or *ABA*-factorizations.

3 ABA-factorizations

The following result was obtained by B. Amberg and L. Kazarin.

Theorem 3.1 *Let* $G = ABA$ *be a finite group with cyclic subgroup* B*. If* A *is abelian or* A *is nilpotent of odd order and* $(|A|, |B|) = 1$*, then* G *is soluble.*

The examples above show that one cannot relax the conditions on A and B considerably.

4 Some arithmetical properties of groups

By famous Brauer and Fowler theorem (On groups of even order. Ann. Math. 62(1955), 565 - 583) every finite simple non-abelian group has a proper subgroup of order at least $|G|^{1/3}$. L. Kazarin and I. Sagirov in 2001 have proved that every finite non abelian simple group G has an irreducible complex character ψ such that $\psi(1) > |G|^{1/3}$. Moreover, if G is different from M_{22} and is not an alternating group, then the group G has 3 irreducible characters ψ_1, ψ_2, ψ_3, not necessary different such that $|G| \mid \psi_1(1)\psi_2(1)\psi_3(1)$. Moreover Sagirov proved that except A_7 and A_{13} this property hold for every simple alternating group A_n with $n \leq 14$. Leter S. Polyakov proved this for $n \leq 96$.

E.P. Wigner in his paper " On representations of nite groups " (Amer.J. Math. 63 (1941), 57-63). has proved the following mysterious inequality

$$\Sigma_{g \epsilon G} |C_G(g)|^2 \geq \Sigma_{g \epsilon G} \zeta(g)^3.$$

Here $\zeta(g)$ is the number of square roots of g in G. The equality holds if and only if G is an SR-group. This means that every element in G is conjugate to its inverse and the tensor product of every two irreducible representations has irreducible components in its decomposition into the sum of irreducible ordinary representations with multiplicities at most 1.

S.P. Strunkov asked whether every finite SR-group is soluble. L.Kazarin and V.Yanishevskii have relaxed the SR-condition to the a question, when the squares of irreducible representations are multiplicity-free and reduced the problem to the case, when the non abelian composition factors of a group,which is a minimal counterexample are isomorphic to groups A_5 and A_6 only. The remaining dicult problem was solved by L. Kazarin and E. Chankov in 2010.

4 Some arithmetical properties of groups

Therefore the following theorem holds:

Theorem 4.1 *Let G be a finite SR-group. Then G is soluble.*

Yanishevskii has described all SR-groups of order at most 2000. The description of supersoluble was obtained by Chankov in the following

Theorem 4.2 *Let G be a finite supersoluble SR-grou with Sylow 2-subgroup S. Then O(G) is an abelian group and G = O(G)S. Moreover, Φ(S) is normal in G and G/Φ(S) is a direct product of a generalized dihedral groups*.

Recentely S.V. Polyakov classied composition factors of finite groups, in which the products of squares of irreducible representations have multiplicities in its decompositions into the sum of irreducible characters at most 2. Also he has found the structural constants in the decompositions of the squares of irreducible characters for simple finite groups.

4 Some arithmetical properties of groups

With the help of Wigner's inequality L. Kazarin and B. Amberg have proved (not using CFSG) the following

Theorem 4.3 Let G be a finite simple group and let μ be its arbitrary involution in G. If $|G| > 2|C_G(\mu)|^3$, then G has a proper subgroup of order at least $|G|^{1/2}$. If $|G| > |C_G(\mu)|^3$, then $|G| < k(G)^3$, where $k(G)$ is the class number of G.

Another bound of the order of a finite simple group gives the following theorem by L. Kazarin:

Theorem 4.4 Let G be a finite simple group and $x \neq 1$ be its arbitrary element. If $m = |G : C_G(x)|$, then $|G| < 2^m$.

4 Some arithmetical properties of groups

The degrees of ordinary irreducible characters are of great interest. In 2008 N. Snyder has studied finite group with an irreducible character of degree d such that $|G| = d(d + e)$. He has proved that the order of G is bounded in terms e, provided $e > 1$. When $e = 1$, Ya. Berkovich proved that G is a Frobenius group with kernel of order $d+1$.

S. Poiseeva and L. Kazarin (Math.Notes 98:2(2015), 237 - 240) studied finite groups with an irreducible character ψ such that $|G| \leq 2\psi(1)^2$. They have proved that in this case every irreducible character of G is a constituent of 2 unless G is an extraspecial 2-group.

In the case when ψ(1) is a power of prime p and G has an abelian Sylow p-subgroup , they obtain a complete description

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Theorem 4.5 *Let G be a finite group with an irreducible character Ψ such that $2\psi(1)^2 \geq |G|$. If $\psi(1) = p^m$ and a Sylow p-subgroup of G is abelian, then either p is a Mersenne prime and G is a direct product of m Frobenius groups of order p(p + 1), or p = 2 and G is a direct product of groups each of which is a Frobenius group of order $q_i 2^{m(i)}$ (where $q_i = 2^{m(i)} + 1$ are Fermat primes) or is a Frobenius group of order $3^2 2^3$ (in this case m(i) = 3), where $\Sigma_i m(i) = m$.*

In general case, the problem of describing the structure of a group having an irreducible character ψ such that $|G| < c\psi(1)^2$ for a small constant *c* seems to be rather complicated. The five Mathieu groups, the Thompson sporadic group and the Janko group of order 604800 have a character of this kind with *c* < 3.1.

V. Zenkov has constructed a group of order $3^3 2^4$ with an irreducible character of degree 24 and non abelian Sylow 2-subgroup. Another results of this kind is possible to find in Abstracts of our Conference.

Remark

This is a short survey, not pretending on the complete description of the theme. For instance, I did not touch the results concerning factorizations of soluble groups (in particular, of *p*-groups). Almost completely are missing the results on infinite groups and completely missing some applications. As acompromise I add some bibliography. Pay an attention to the survey papers here and a paper of O. H. Kegel in [12].

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