ON PRIME GRAPHS OF FINITE GROUPS

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I shall give a survey of some results on the above-mentioned topic obtained by the author jointly with his students, post-graduates or post-doctors.

The study of finite groups depending on their arithmetical properties (orders of elements and subgroups, sizes of conjugacy classes, various π-properties, degrees of irreducible characters and so on) is an important direction in finite group theory having rich history. The classification of finite simple groups reduces often this study to the case of *almost simple groups*, i. e., groups A such that $Inn(P) \leq A \leq Aut(P)$ for a finite simple non-abelian group P.

Let G be a finite group. Denote by $\omega(G)$ the set of all element orders (the *spectrum*) of G and by $\pi(G)$ the set $\pi(G)$ of all prime divisors of $|G|$ (the *prime spectrum*) of G. The set $\omega(G)$ determines the *prime graph* (or the *Gruenberg—Kegel graph*) $\Gamma(G) = GK(G)$ of G, in which the vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \omega(G)$. The graph $\Gamma(G)$ can be considered as a subset of the spectrum $\omega(G)$ consisting of all products of two distinct primes from $\omega(G)$.

Let $s = s(G)$ be the number of connected components of $GK(G)$ and let $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$ be the set of connected components of $GK(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1(G)$.

The notion of prime graph appeared by the investigation some cohomological problems related to integer representations of finite groups and was found very fruitful. It is interesting that the prime graph $\Gamma(G)$, in contrast to the spectrum $\omega(G)$, can be determined by the character table of the group G. As Isaacs observed, the well-known commuting graph $\Gamma(G, G \setminus \{1\})$ of a non-abelian group G with $Z(G) = 1$ on the set $G \setminus \{1\}$ determines also the prime graph $\Gamma(G)$. Therefore, the prime graph of a finite group is its fundamental arithmetical invariant, having numerous applications. An interesting general problem arises: *describe all finite groups whose prime graph has a given property.* The problems of the recognizability of finite groups by spectrum or by prime graph are particular cases of such problem.

A finite group G is said to be *recognizable* by spectrum (resp. by prime graph), if for any finite group H with $\omega(H) = \omega(G)$ (resp. $\Gamma(H) = \Gamma(G)$) we have $H \cong G$.

In the middle 1980th, Shi made first steps in the solving the problem of the recognizability of finite simple groups by spectrum. A big progress in the solving this problem is obtained today. So, it is practically reduced to the case of almost simple groups.

We will consider some recent results of the study of finite groups by the properties of their prime graphs only.

The first result about finite groups with disconnected prime graph is the following structural theorem obtained by Gruenberg and Kegel about 1975 in an unpublished paper (the proof of this theorem was published in the paper of Williams (J. Algebra, 1981), post-graduate of Gruenberg.

Theorem 1 (Gruenberg—Kegel theorem). *If G is a finite group with disconnected prime graph, then one of the following holds:*

(a) *G is a Frobenius group;*

(b) *G is a 2-Frobenius group, i. e., $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius groups with cores A and B and complements B and C, respectively;*

(c) *G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $Inn(P) \leq A \leq Aut(P)$, P is a finite simple group with $s(G) \leq s(P)$, and A/P is a $\pi_1(G)$-group.*

This theorem implies the complete description of solvable finite group with disconnected prime graph (they are groups from items (a) or (b) of the theorem).

Moreover, as Theorem 1 shows, the question of the studying a non-solvable finite group with disconnected prime graph is reduced largely to the studying some properties of simple non-abelian groups.

Williams (J. Algebra, 1981) obtained an explicit description of connected components of the prime graph for all finite simple non-abelian groups except the groups of Lie type of even characteristic.

AK (Mat. Sbornik, 1989) obtained such description for the remaining case of the groups of Lie type of even characteristic.

Later this result was repeated by Iiyori and Yamaki (J. Algebra, 1993) in connection with an application of prime graph to the proof of well-known Frobenius conjecture.

But later, some inaccuracies in all three papers were found. So, Suzuki indicated on a mistake in the paper of Iiyori and Yamaki. They (J. Algebra, 1996) corrected the mistake, but some their other inaccuracies are remained.

In a joint work of Mazurov and myself (Siberian Math. J., 2000), the corresponding tables were corrected.

According to Suzuki, a proper subgroup H of a group G is called *isolated subgroup (or CC-subgroup)* in G if $C_G(h) \leq H$ for all non-trivial element and called *TI-subgroup* in G if the intersection $H \cap H^g$ is equal to 1 or H for all $g \in G$. It is easily to understand that an isolated subgroup H in a finite group G is $\pi(H)$-Hall subgroup in G. It is well-known that the core and the complement in a Frobenius group are its isolated subgroups.

Finite groups having an isolated subgroup are studied without the classification of finite simple groups by many known algebraists (Frobenius, Suzuki, Feit, Thompson, G. Higman, Arad, Chillag, Busarkin, Gorchakov, Podufalov and others).

Williams (J. Algebra, 1981) established a relation between connected components of the graph $\Gamma(G)$ and isolated subgroups of odd order in finite non-solvable group G.

Theorem 2. *If G is a finite non-solvable group with disconnected prime graph then, for any $i > 1$, the group G contains a nilpotent isolated $\pi_i(G)$-Hall TI-subgroup $X_i(G)$.*

Theorems 1 and 2 imply that the class of finite groups with disconnected prime graph coincides with the class of finite groups having an isolated subgroup.

If G is a finite simple group then these subgroups $X_i(G)$ $(i > 1)$ are abelian and are determined to within the conjugacy. Moreover, the isomorphic types of $X_i(G)$ for $i > 1$ are also determined.

The classification of connected components of prime graph for finite simple groups were applied by Lucido (Rend. Sem. Mat. Univ. Padova, 1999, 2002) for obtaining analogous classification for all finite almost simple groups.

Our attention draws a more detailed study of the class of finite groups with disconnected prime graph.

The finite simple groups with disconnected prime graph compose sufficiently restricted class of all finite simple groups, but include many “small” in various senses groups which arise often in the investigations. For example, all finite simple groups of exceptional Lie type besides the the groups $E_7(q)$ for $q > 3$, as well as simple groups from the well-known "Atlas of finite groups" besides the group A_{10}, have disconnected prime graphs.

The following natural problem arises.

Problem 1. *Study the finite non-solvable groups with disconnected prime graph, which are not almost simple.*

Problem 1 is solved for several particular cases only, because here some non-trivial problems related with modular representations of finite almost simple groups arise. Let us consider such a problem.

Let G be a finite group with disconnected prime graph, and let G be nonisomorphic to a Frobenius group or a 2-Frobenius group. Then, by the Gruenberg–Kegel theorem, the group $\overline{G} := G/F(G)$ is almost simple and is known by the above mentioned results. Assume that $F(G) \neq 1$. Each connected component $\pi_i(G)$ of the graph $\Gamma(G)$ for $i > 1$ corresponds to a nilpotent isolated $\pi_i(G)$-Hall subgroup $X_i(G)$ of the group G. Any nontrivial element x from $X_i(G)$ $(i > 1)$ acts *fixed-point-freely (freely)* on $F(G)$, i. e., $C_{F(G)}(x^n) = 1$ for all $x^n \neq 1$. Let K and L be two neighboring terms of a chief series of the group G and $K < L \leq F(G)$). Then, the (chief) factor $V = L/K$ is an elementary abelian p-group for some prime p (we will call it the *p-chief factor* of the group G), and we can consider it as a faithful irreducible $GF(p)\overline{G}$-module (since $C_{G/K}(V) = F(G)/K$). Moreover, any nontrivial element from $X_i(G)$ $(i > 1)$ acts fixed-point-freely on V.

Therefore, the problem of studying the structure of the group G largely reduces to the following problem, which is of independent interest.

Problem 2. *For the finite almost simple group G and given prime p, describe all irreducible $GF(p)G$-modules V such that an element of prime order ($\neq p$) from G acts on V fixed-point-freely.*

Results on Problem 2 have numerous applications, in particular, for the study of finite groups by the properties of their prime graphs.

Extending and refining Problem 1 we obtain the following

Problem 3. *Let G be a finite group, Q be a normal nontrivial subgroup from G, $\overline{G} = G/Q$ be a known group and an element of prime order from $G \setminus Q$ acts on Q fixed-points-freely. The following questions arise.*

1) *What are the chief factors of the group G in Q as $\overline{G}$-modules?*

2) *What is the structure of the group Q (isomorphic type, nilpotency class, exponent, derived length etc.)?*

3) *If Q is elementary abelian group, is the action of $\overline{G}$ on Q completely irreducible?*

4) *Is the extension of G over Q splittable?*

The well-known Thompson's theorem (1959) implies that Q is a nilpotent group in this situation. Results on the item 1) of Problem 3 are used for the solving the remaining items of this problem. Moreover, they are useful for the study of the reconizability of finite simple groups by spectrum or prime graph.

Problem 3 for finite non-solvable groups can be considered as an extension of Mazurov's problem 17.72 from "Kourovka notebook" about 2-Frobenius groups.

In spite of importance of the questions 1) — 4), we have a few results about them. In general, this important problem is far from being solved.

The first work, devoted to the study of the case when $\overline{G}$ is a simple nonabelian group, was a classical work of G. Higman (lecture notes, 1968). If $\overline{G} \cong L_2(2^m)$ for $m \geq 2$ and an element of order 3 from G acts on Q fixed-point-freely then Higman shows that Q is an elementary abelian 2-group, the action of $\overline{G}$ on Q is completely irreducible and every 2-chief factors of G is isomorphic to the natural $GF(2^m)SL_2(2^m)$-module.

Later Martineau (J. London Math. Soc. (2), 1972; Amer. J. Math. Soc., 1972) obtained an analogous result for the case when $\overline{G}$ is isomorphic to the Suzuki group $Sz(2^n)$ and an element of order 5 from G acts on Q fixed-point-freely.

Continuing the work of Higman, Stewart (Proc. London Math. Soc., 1973) showed that $Q = 1$ in the case when $\overline{G} \cong L_2(q)$ for odd $q > 5$ and an element of order 3 from G acts on Q fixed-point-freely.

The papers of Prince (J. Algebra, 1977; Proc. Roy. Soc. Edinburgh. Sect A, 1982), Zurek (Mitt. Math. Sem. Giessen, 1982), Holt and Plesken (Quart. J. Math. Oxford. Ser. 2, 1986) were devoted to the study of the case, when $Q = O_2(G)$, $\overline{G} \cong A_5$ and an element of order 5 from G acts on Q fixed-point-freely. This case is difficult, because in the case Q can be nonabelian group. Prince and Zurek gave affirmative answers on the questions 1), 3) and 4). In particular, Q is a product of $\overline{G}$-invariant subgroups Q_i's, isomorphic to either a homocyclic 2-group of the rank 4, or the special 2-group of order 2^8 with the center of order 2^4 (isomorphic to the unipotent radical some parabolic maximal subgroup in $U_5(2)$). In addition, in the first case every 2-chief factor of G involving in Q_i is isomorphic to the orthogonal (permutational) $GF(2)A_5$-module, and in the second case the group $Z(Q_i)$ is isomorphic to the orthogonal $GF(2)A_5$-module, but $Q_i/Z(Q_i)$ is isomorphic to the natural $GF(4)SL_2(4)$-module. By an early result of G. Higman (J. London Math. Soc. (2), 1957), a theoretical upper bound of the nilpotency class of Q was 6. Zurek conjectured that such bound must be 2. But later on, Holt and Plesken proved that the nilpotency class of Q is at the most 3 and constructed an example of the group Q of order 2^{28} where this bound is reached. Using a computer, they showed also that this is an example of minimal order.

Prince (J. Algebra, 1977; Proc. Roy. Soc. Edinburgh. Sect A, 1982) proved that if $Q = O_2(G)$, $\overline{G} \cong A_6$ and an element of order 5 from G acts on Q fixed-point-freely then the questions 1) – 4) are solved affirmatively.

Dolfi, Jabara and Lucido (Siberian Math. J., 2004) in the frame of the classification of $C55$-groups proved that if $\overline{G} \cong A_6$ and an element of order 5 from G acts on Q fixed-point-freely, then $O(Q)$ is abelian, $O(Q) = O_3(G)$ and 3-chief factors of G are isomorphic to the 4-dimentional permutational $GF(3)\overline{G}$-module. In this paper, it is asserted also that if $\overline{G} \cong A_5$ and an element of order 5 from G acts on Q fixed-point-freely, then $O(Q)$ is abelian. But this assertion is found wrong. Recently, Astill, Parker and Waldecker (Siberian Math. J., 2012) proved that in this situation $O(Q)$ is a nilpotent group of class at most 2 and, for any odd prime $p \neq 5$, constructed a r-group of class 2 admitting the group A_5 with the mentioned property. Moreover, in the last paper, the questions 1) – 4) are solved affirmatively for the cases when an element of order 5 from G acts on Q fixed-point-freely and either $Q = O_3(G)$ and $\overline{G} \cong A_6$ or $Q = O_7(G)$ and $\overline{G} \cong L_2(49)$. These results refined the classification of $C55$-groups, obtained by Dolfi, Jabara and Lucido.

If the socle of the group $\overline{G}$ is a finite simple group of Lie type over a field of a prime characteristic p, then, for the solving the item 1) of Problem 2, the classification of Guralnick and Tiep (J. Group Theory, 2003) of all unisingular finite simple group of Lie type is useful. A finite simple group X of Lie type over a field of a prime characteristic p is called *unisingular* if any element $s \in X$ has a non-trivial fixed point in any non-trivial finite abelian p-group on which X acts.

Zavarnitsine (Sib. Math. J., 2008; Sib. Electron. Mat. Izv., 2011) found some sufficient conditions for an element of a large prime order in the group $S = L_n^{\pm}(q)$, where q is a power of a prime p, to have non-zero fixed points in S-modules over a field of characteristic p.

The following problem is natural.

Problem 4. *Let $G = SL_n(q)$, where q is a power of a prime p, and S be a Singer cycle of G, i. e., a cyclic subgroup of order $(q^n - 1)/(q - 1)$. Classify absolutely irreducible G-module in characteristic p on which an element of prime order r from S acts fixed-point-freely. The natural irreducible $GF(q)G$-module is such for all non-trivial elements of S.*

Note that if H is a finite group with disconnected prime graph such that $F(H) \neq 1$ and $\overline{H} = H/F(H) \cong PSL_n(q)$ for $n \geq 3$ then the action (by conjugacy) of H on $F(H)$ induces on each chief factor of H involving to $F(H)$ a faithful irreducible $\overline{H}$-module (for some field of prime order) in which all non-trivial elements from a Singer cycle of the group $\overline{H}$ act fixed-point-freely. It is very important case of Problem 1.

AK, Suprunenko and Osinovskaya (Trudy Inst. Mat. Mekh., 2013) solved Problem 4 for the case when the residue modulo r of the number q generates the multiplicative group of the field $GF(r)$ or $r \in \{3, 5\}$. This result generalizes, in particular, the well-known results by G. Higman (1968) and Stewart (1973) which are obtained in the case when $r = 3$ and $n = 2$.

The considered partial results show that Problem 2 is complicated. In general, this important problem is far from being solved.

If the table of irreducible Brauer characters is known (for example, from "An atlas of Brauer characters", 1995 or GAP) then the following known results can be applied for the solving the item 1) of Problem 2.

Proposition 1. *Suppose that G is a finite quasi-simple group, F is a field of characteristic $p > 0$, V is a faithful absolutely irreducible FG-module, and β is a Brauer character of the module V. If g is an element in G of a prime order coprime to $p|Z(G)|$, then*

$$dim\ C_V(g) = (\beta|_{\langle g\rangle}, 1|_{\langle g\rangle}) = \frac{1}{|g|}\sum_{x\in\langle g\rangle}\beta(x).$$

Proposition 2. *Let G be a finite group, $F = GF(p^m)$ the field of definition of characterictic p for (absolutely) irreducible FG-module V, $\langle\sigma\rangle = Aut(F)$, V_0 denote module V regarded as $GF(p)G$-module, and $W = V_0^F$. Then the following holds:*

(1) *$W = \bigoplus_{i=1}^m V^{\sigma^i}$, where V^{σ^i} is the module algebraically conjugated to V by means of σ^i;*

(2) *V_0 is an irreducible $GF(p)G$-module and, in particular, W is realized as irreducible $GF(p)G$-module V_0;*

(3) *irreducible $GF(p)G$-modules are found in a bijective correspondence (up to isomorphism of modules) with the classes of algebraic conjugacy of irreducible $\overline{GF(p)}G$-modules (V_0 corresponds to the class $\{V^{\sigma^i} \mid 1 \le i \le m\}$).*

In the frame of above-mentioned Problem 1, we investigate finite groups whose prime graph is disconnected and has a small number of vertices.

A finite group G is called *n-primary* if $|\pi(G)| = n$. First of all, let us consider the trivial cases, when the prime graph of a finite group has one or two vertices.

The class of 1-primary groups coincides with the boundless class of all primary groups. Using Gruenberg-Kegel theorem and the properties of solvable complements in finite Frobenius groups, it is not difficult to describe 2-primary (biprimary) groups with disconnected prime graph. They are either Frobenius groups or 2-Frobenius groups of a special form.

AK and his postgraduate Khramtsov (Trudy Inst. Mat. Mekh. UrO RAN, 2010; Sib. Electron. Mat. Izv., 2012) described the chief factors of 3-primary groups with disconnected prime graph. In particular, the following theorem is proved.

Theorem 3. *Let G be a finite 3-group with disconnected prime graph and $\overline{G} = G/F(G)$. Then, one of the following statements holds*:

(1) *G is a Frobenius group.*

(2) *G is a 2-Frobenius group.*

(3) *$s(G) = 3$ and either G is isomorphic to A_5, A_6, $L_2(7)$, $L_2(8)$, M_{10} or $L_2(17)$, or $G/O_2(G) \cong L_2(2^n)$, where $n \in \{2, 3\}$ and $O_2(G)$ is a direct product of minimal normal subgroups of the order 2^{2n} in G, each of which as $\overline{G}$-module is isomorphic to the natural $GF(2^n)SL_2(2^n)$-module.*

(4) *$s(G) = 2$, $\pi_1(G) = \{2, 5\}$ and $G \cong PGL_2(9)$.*

(5) *$s(G) = 2$, $\pi_1(G) = \{2, 3\}$, $F(G) = O_2(G) \times O_3(G)$, and one of the following statements $(i) - (viii)$ holds:*

(i) $\overline{G} \cong A_5$ or S_5, any 2-chief factor of the group G as $GF(2)\overline{G}$-module is isomorphic to one of two 4-dimensional irreducible $GF(2)\overline{G}$-module, any 3-chief factor of G as $GF(3)\overline{G}$-module is isomorphic to the 4-dimensional irreducible permutation $GF(3)\overline{G}$-module.

(ii) $\overline{G} \cong A_6$, S_6 or M_{10}, $F(G)$ is the direct product of an elementary abelian 2-group and an abelian 3-group, and $F(G) \neq 1$ for $\overline{G} \cong A_6$ or M_{10}. If $O_2(G) \neq 1$ then $O_2(G)$ is the direct product of G'-invariant subgroups of order 16 that are as $GF(2)\overline{G}'$-module isomorphic to either the 4-dimensional irreducible permutation $GF(2)A_6$-module or conjugated with them by an outer automorphism of S_6. Any 3-chief factor in G' as $GF(3)\overline{G}'$-module is isomorphic to the 4-dimensional irreducible permutation $GF(3)A_6$-module.

(iii) $\overline{G} \cong U_4(2)$ and $F(G) = O_2(G)$ is an elementary abelian 2-group.

Any 2-chief factor of the group G as $GF(4)\overline{G}$-module is isomorphic to the natural unitary 4-dimensional $GF(4)SU_4(2)$-module.

(iv) $\overline{G} \cong L_2(8)$ or $Aut(L_2(8))$, $F(G) = O_2(G)$, and $F(G) \neq 1$ for $\overline{G} \cong L_2(8)$. Any 2-chief factor of the group G' as $GF(8)\overline{G}'$-module is isomorphic to the natural 2-dimensional $GF(8)SL_2(8)$-module or 4-dimensional irreducible $GF(8)L_2(8)$-module.

(v) $\overline{G} \cong L_2(7)$ or $PGL_2(7)$, and $F(G) \neq 1$ for $\overline{G} \cong L_2(7)$. Any 2-chief factor of the group G' as $GF(2)\overline{G}'$-module is isomorphic to the natural 3-dimensional $GF(2)SL_3(2)$ modul or to the module conjugated with them by an outer involutive automorphism of the group $SL_3(2)$. Any 3-chief factor of the group G' as $\overline{G}'$-module is isomorphic to the 3-dimensional irreducible $GF(9)L_2(7)$-module or the 6-dimensional absolutely irreducible $GF(3)L_2(7)$-module.

(vi) $\overline{G} \cong U_3(3)$ or $Aut(U_3(3))(\cong G_2(2))$. Any 2-chief factor of the group G as $GF(2)\overline{G}$-module is isomorphic to the 6-dimensional absolutely irreducible $GF(2)\overline{G}$-module. Any 3-chief factor of the group G' as $GF(9)\overline{G}'$-module is isomorphic to the natural unitary 3-dimensional $GF(9)U_3(3)$-module or the 6-dimensional $GF(9)U_3(3)$-module.

(vii) $\overline{G} \cong L_3(3)$ or $Aut(L_3(3))$. Any 2-chief factor of the group G' as $GF(2)\overline{G}'$-module is isomorphic to the 12-dimensional absolutely irreducible $GF(2)L_3(3)$-module. Any 3-chief factor of the group G' as $GF(3)\overline{G}'$-module is isomorphic to one of the three absolutely irreducible $GF(3)L_3(3)$-modules of the dimensions 3, 6 or 15; for those dimensions up to isomorphism there exists exactly two $GF(3)L_3(3)$-modules that are conjugated by an outer involutive automorphism of the group $L_3(3)$.

(viii) $\overline{G} \cong L_2(17)$ or $PGL_2(17)$ and $F(G) \neq 1$ for $\overline{G} \cong L_2(17)$. Any 2-chief factor of the group G' as $\overline{G}'$-module is isomorphic either to the 8-dimensional absolutely irreducible $GF(2)L_2(17)$-module, to the module conjugated with them by an outer involutive automorphism of the group $L_2(17)$, to the 16-dimensional absolutely irreducible $GF(2)L_2(17)$-module, or to the 16-dimensional irreducible $GF(8)L_2(17)$-module. Any 3-chief factor of the group G as $GF(3)\overline{G}$-module is isomorphic to the 16-dimensional absolutely irreducible $GF(3)\overline{G}$- module.

Each item of the theorem is realized.

The proof of Theorem 7 uses the well-known description of finite simple 3-primary groups (see, for example, Herzog (J. Algebra, 1968)).

As a corollary of Theorem 5, the following result is obtained.

Theorem 4. *The finite* 3*-primary almost simple group with disconnected prime graph is recognizable by prime graph if and only if it is isomorphic to* $L_2(17)$.

AK and Khramtsov (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2011) described chief factors of commutator subgroups of finite 4-primary groups with disconnected prime graph. In some cases, all possibilities for such chief factors were not determined; however, the existence of at least one possibility was proved. The description is too large so we formulate here only first theorem, which was proved.

Theorem 5. *Let* G *be a finite* 4*-primary group with disconnected prime graph, and let* $\overline{G} = G/F(G)$. *Then, one of the following statements holds:*

(1) G *is a Frobenius group;*

(2) G *is a* 2*-Frobenius group;*

(3) $\overline{G}$ *is an almost simple triprimary group;*

(4) $\overline{G} \cong L_2(2^m)$, *where* $m \geq 5$, $2^m - 1$, *and* $(2^m + 1)/3$ *are primes;*

(5) $\overline{G} \cong L_2(3^m)$ *or* $PGL_2(3^m)$, *where* m *and* $(3^m - 1)/2$ *are odd primes and* $(3^m + 1)/4$ *is either a prime or* 11^2 *(for* $m = 5$*);*

(6) $\overline{G} \cong L_2(r)$ *or* $PGL_2(r)$, *where* r *is a prime,* $17 \neq r \geq 11$, $r^2 - 1 = 2^a 3^b s^c$, $s > 3$ *is a prime,* $a, b \in \mathbb{N}$, *and* c *is either* 1 *or* 2 *for* $r \in \{97, 577\}$;

(7) $\overline{G} \cong A_7$, S_7, A_8, S_8, A_9, $L_2(16)$, $L_2(16){:}\,2$, $\mathrm{Aut}(L_2(16))$, $L_2(25)$, $L_2(25){:}\,2$, $L_2(27){:}\,3$, $L_2(49)$, $L_2(49){:}\,2_1$, $L_2(49){:}\,2_3$, $L_2(81)$, $L_2(81){:}\,2$, $L_2(81){:}\,4$, $L_3(4)$, $L_3(4){:}\,2_1$, $L_3(4){:}\,2_3$, $L_3(5)$, $\mathrm{Aut}(L_3(5))$, $L_3(7)$, $L_3(7){:}\,2$, $L_3(8)$, $L_3(8){:}\,2$, $L_3(8){:}\,3$, $\mathrm{Aut}(L_3(8))$, $L_3(17)$, $\mathrm{Aut}(L_3(17))$, $L_4(3)$, $L_4(3){:}\,2_2$, $L_4(3){:}\,2_3$, $U_3(4)$, $U_3(4){:}\,2$, $\mathrm{Aut}(U_3(4))$, $U_3(5)$, $U_3(5){:}\,2$, $U_3(7)$, $\mathrm{Aut}(U_3(7))$, $U_3(8)$, $U_3(8){:}\,2$, $U_3(8){:}\,3_1$, $U_3(8){:}\,3_3$, $U_3(8){:}\,6$, $U_3(9)$, $U_3(9){:}\,2$, $\mathrm{Aut}(U_3(9))$, $U_4(3)$, $U_4(3){:}\,2_2$, $U_4(3){:}\,2_3$, $U_5(2)$, $\mathrm{Aut}(U_5(2))$, $S_4(4)$, $S_4(4){:}\,2$, $\mathrm{Aut}(S_4(4))$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_4(9){:}\,2_1$, $S_4(9){:}\,2_3$, $S_6(2)$, $G_2(3)$, $\mathrm{Aut}(G_2(3))$, $O_8^+(2)$, ${}^3D_4(2)$, $\mathrm{Aut}({}^3D_4(2))$, $Sz(8)$, $Sz(32)$, $\mathrm{Aut}(Sz(32))$, ${}^2F_4(2)'$, ${}^2F_4(2)$, M_{11}, M_{12}, $\mathrm{Aut}(M_{12})$, *or* J_2.

The proof of Theorem 5 uses the description of finite simple 4-primary groups obtained by Shi (Chinese Science Bull., 1991), Huppert and Lempken (Proc. F. Scorina. Gomel State University. Problems in Algebra, 2000) and Bugeaud, Cao and Mignotte (J. Algebra, 2001). Shi wrote Question 13.65 in "The Kourovka Notebook": *is the number of finite simple tetraprimary groups finite or infinite?* However, Shi's question is still open.

In the proofs of Theorems from (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2010, 2011; Sib. Electron. Mat. Izv., 2012), computations are carried out by applying the computer system GAP. A program written in the language of this system makes it possible to compute by the formula from Proposition 1 the dimension of the centralizer in the vector space of an element of prime order from a finite simple group that acts irreducibly on this space.

As a corollary of Theorems 1–8 from (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2011), the following result is obtained.

Theorem 6. *A finite* 4*-primary simple group is recognizable by prime graph if and only if it is isomorphic to one of the following groups:* A_8*,* $L_3(4)$*, and* $L_2(q)$*, where* $|\pi(q^2-1)| = 3$*,* $q > 17$*, and either* $q = 3^m$ *and* m *is an odd prime or* q *is a prime and* $q \not\equiv 1 \pmod{12}$ *or* $q \in \{97, 577\}$*.*

Theorems 5 and 6 from (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2011) that are concerned with 4-primary sporadic groups M_{11}, M_{12}, and J_2 refine essentially the corresponding Hagie's results (Commun. Algebra, 2003).

Vasil'ev wrote Problem 16.26 in "The Kourovka Notebook" about the finding the maximal number of pairwise nonisomorphic finite nonabelian simple groups with the same prime graph. There is conjecture that this number equals to 5 and is achieved on the groups J_2, A_9, $C_3(2)$, $D_4(2)$. Theorem 6 from (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2011) shows that the set $\{J_2, A_9, C_3(2), D_4(2)\}$ is a maximal set of pairwise nonisomorphic finite nonabelian simple groups with the same prime graph.

AK and Khramtsov (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2012) obtained the positive solution for all items of Problem 3 in the case when $Q = O_2(G)$, $\overline{G} \cong A_7$ and an element of order 5 from G acts on Q fixed points freely. The following theorem is proved.

Theorem 7. *Let G be a finite group with a nontrivial normal 2-subgroup Q and $G/Q \cong A_7$. Suppose that an element of order 5 from G acts on Q fixed points freely. Then the extension G over Q is split, Q is an elementary abelian group and Q is the direct product of minimal normal subgroups each of which as $GF(2)G/Q$-module is isomorphic to one of the two 4-dimensional irreducible $GF(2)A_7$-modules that are conjugated by outer automorphism of the group A_7.*

In several recent works of AK, Khramtsov, Suprunenko, Kolpakova (2014-2015), a description of chief factors of 4-primary groups with disconnected prime graph was corrected and refined.

B. Khosravi (2009) obtained a description of a group having the same prime graph as the group $Aut(S)$ for any sporadic simple group S except for the group J_2. He posed the problem: *describe all groups G such that* $\Gamma(G) = \Gamma(Aut(J_2))$. Note that if S is a sporadic simple group then $|Aut(S) : S| \leq 2$ and graphs $\Gamma(S)$ and $\Gamma(Aut(S))$ are disconnected except for the graphs $\Gamma(Aut(J_2))$ and $\Gamma(Aut(McL))$. AK (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2012) solved the Khosravi's problem. The following theorem is proved.

Theorem 8. *Let G be a finite group, $\Gamma(G) = \Gamma(Aut(J_2))$ and $\overline{G} = G/O_2(G)$. Then one of the following statements holds:*

(1) *G is soluble, 2-complement in G is a Frobenius group, whose core is a 7-group and complement B is a cyclic $\{3,5\}$-group of order divisible on 15, the factor-group $G/O_{\{2,7\}}(G)$ is isomorphic to a subgroup of order dividing $8|B|$ from $Hol(C)$;*

(2) *G is soluble, 2-complement R in G is a Frobenius group of form $A : B$, where $A = F(R)$ is a biprimary $\{3,5\}$-group, and B is a cyclic 7-group, the factor-group $O_{7'}(G)/O_2(G)$ has the normal 2-complement $AO_2(G)/O_2(G)$, and the factor-group $G/O_{7'}(G)$ is isomorphic to B or the dihedral group of order $2|B|$;*

(3) *G is soluble, 2-complement R in G is a 2-Frobenius group of form $A : B : C$, where $A = F(R)$ is a $\{3,5\}$-group of order divisible on 5, B is a cyclic 7-group, and $|C| = 3$, the factor-group $O_{7'}(G)/O_2(G)$ has the normal 2-complement $AO_2(G)/O_2(G)$, and the factor-group $G/O_{7'}(G)$ is isomorphic to a Frobenius group of order $3|B|$ or $6|B|$;*

(4) *$\overline{G}$ is isomorphic to one of the groups A_8, S_8, A_9, S_9, $S_6(2)$, $O_8^+(2)$, $O_8^+(2) : 2$, J_2 or $Aut(J_2)$;*

(5) *$\overline{G}$ is isomorphic to an extension of a nontrivial nilpotent $\{3,5\}$-group A by a group B such that $F^*(B) = O_2(B) \times L$, where the group L is isomorphic to A_7, the group $B/O_2(B)$ is isomorphic to A_7 or S_7, the group L induces (by conjugation) on any p-chief factor of the group $\overline{G}^\infty$ the irreducible 6-dimensional $GF(p)A_7$-module for $p \in \{3,5\}$;*

(6) *$\overline{G}$ is isomorphic to an extension of a nilpotent $\{3,5\}$-group A of order divisible on 5 by a group B such that $F^*(B) = O_2(B) \times L$, where the group L is isomorphic to $U_3(3)$, the group $B/O_2(B)$ is isomorphic to $U_3(3)$ or $G_2(2)$, the group L induces on any 3-chief factor of the group $\overline{G}^\infty$ the natural unitary 3-dimensional $GF(9)U_3(3)$-module or the irreducible 6-dimensional $GF(9)U_3(3)$-module, and on any its 5-chief factor the absolutely irreducible 6-dimensional $GF(5)U_3(3)$-module;*

(7) $\overline{G}$ *is isomorphic to an extension of a nilpotent* $\{3,5\}$*-group* A *of order divisible on* 5 *by a group* B *such that* $F^*(B) = O_2(B) \times L$*, where the group* L *is isomorphic to* $L_2(7)$*, the group* $B/O_2(B)$ *is isomorphic to* $L_2(7)$ *or* $PGL_2(7)$*, the group* L *induces on any* p*-chief factor of the group* $\overline{G}^{\infty}$ *the irreducible* 3*-dimensional* $GF(p^2)L_2(7)$*-module or the absolutely irreducible* 6*-dimensional* $GF(p)L_2(7)$*-module for* $p \in \{3,5\}$*;*

(8) $\overline{G}$ *is isomorphic to a semidirect product of a nontrivial abelian* 3*-group* A *on a group* B *such that* $F^*(B) = O_2(B) \circ L$*, where the group* L *is isomorphic to* $2{\cdot}L_3(4)$ *or* $2{\cdot}U_4(3)$*, the group* $B/F^*(B)$ *is isomorphic to a subgroup from* D_8*, the involution from* $Z(L)$ *inverts* A*, and the group* L *induces on any* 3*-chief factor of the group* AL *the faithful irreducible* 6*-dimensional* $GF(3)L$*-module;*

(9) $\overline{G}$ *is isomorphic to a semidirect product of a nontrivial abelian* 3*-group* A *on a group* B *such that* $F^*(B) = O_2(B) \circ L$*, where the group* L *is isomorphic to* $2^{2}{\cdot}L_3(4)$*, the group* $B/F^*(B)$ *is isomorphic to a subgroup from* 2^2*,* $Z(L)$ *is generated by some involutions* z_1 *and* z_2 *such that* $A = C_A(z_1) \times C_A(z_2)$*, and the group* L *induces on any* 3*-chief factor of the group* AL *the faithful irreducible* 6*-dimensional* $GF(3)2{\cdot}L_3(4)$*-module;*

(10) $\overline{G}$ *is isomorphic to a semidirect product of a abelian* $\{3,5\}$*-group* A *on a group* B *such that* $F^*(B) = O_2(B) \circ L$*, where the group* L *is isomorphic to* $2{\cdot}J_2$*, the group* $B/O_2(B)$ *is isomorphic to* J_2 *or* $Aut(J_2)$*, the involution from* $Z(L)$ *inverts* A*, and the group* L *induces on any* 3*-chief factor of the group* AL *the faithful irreducible* 6*-dimensional* $GF(9)L$*-module and on any its* 5*-chief factor the faithful irreducible* 6*-dimensional* $GF(5)L$*-module;*

(11) $\overline{G}$ *is isomorphic to an extension of a nilpotent* $\{3,5\}$*-group* A *of order divisible on* 5 *by a group* B *such that* $F^*(B) = O_2(B) \circ L$*, where the group* L *is isomorphic to* $SL_2(7)$*, the group* $B/O_2(B)$ *is isomorphic to* $L_2(7)$ *or* $PGL_2(7)$*, and the group* L *induces on any* p*-chief factor of the group* $\overline{G}^{\infty}$ *for* $p \in \{3,5\}$ *either a unfaithful irreducible* L*-module with the core of order* 2 *(see item* (7)*), or the faithful irreducible* 6*-dimensional* $GF(p^2)L$*-module.*

Each from items (1)–(11) *of the theorem is realised.*

As a corrolary of Theorem 8, we obtain

Corollary. A finite group G such that $|G| = |Aut(J_2)|$ and $\Gamma(G) = \Gamma(Aut(J_2))$ is isomorphic to $Aut(J_2)$, $2 \times J_2$ or $2{\cdot}J_2$.

The group A_{10} is exceptional in many senses. It is the only group with connected prime graph among all finite simple groups from "Atlas of finite groups" and also among all 4-primary simple groups. The non-recognizability by spectrum of the group A_{10} is established by Mazurov yet in 1998. Staroletov (2008, 2010) determined the structure of the group G such that $\omega(G) = \omega(A_{10})$ and in particular proved its unsolvability. Moghaddamfar (2010) proved that the group A_{10} is recognizable by its prime graph and order. Extending these results, AK (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2013) described all finite group with the same prime graph as the group A_{10}. The following theorem is proved.

Theorem 9. *Let G be a finite group, $\Gamma(G) = \Gamma(A_{10}$ and $\overline{G} = G/O_3(G)$. Then one of the following statements holds:*

(1) *G is soluble, 3-complement R in G is a Frobenius group, whose core is a non-cyclic 7-group and complement B is a biprimary group of form $C : D$, where C is a cyclic 5-group and D is a cyclic or (generalized) quaternion 2-group, the factor-group $G/O(G)$ is isomorphic to D, $SL_2(3)$, or $Q_8.S_3$;*

(2) *G is soluble, 3-complement R in G is a Frobenius group of form $A : B$, where $A = F(R)$ is a biprimary $\{2, 5\}$-group and B is a cyclic 7-group, the factor-group $O_{7'}(G)/O_3(G)$ has the normal 3-complement $AO_3(G)/O_3(G)$, and the factor-group $G/O_{7'}(G)$ is isomorphic to B or a Frobenius group of order $3|B|$;*

(3) *G is soluble, 3-complement R in G is a 2-Frobenius group of form $A : B : C$, where $A = F(R)$ is a $\{2, 5\}$-group of order divisible on 5, B is a cyclic 7-group, and $|C| = 2$, the factor-group $O_{7'}(G)/O_3(G)$ has the normal 2-complement $AO_3(G)/O_3(G)$, and the factor-group $G/O_{7'}(G)$ is isomorphic to a Frobenius group of order $2|B|$ or $6|B|$;*

(4) *$\overline{G}$ is isomorphic to a semidirect product of a nontrivial abelian 7-group A on a group B such that $F^*(B) = O_3(B) \times L$, where the group L is isomorphic to $SL_2(q)$ for $q \in \{5, 9\}$, the group $B/O_3(B)$ is isomorphic to $L_2(q)$ or $PGL_2(q)$, and any 7-chief factor of the group AL as $L-$module is isomorphic for $q = 5$ to the faithful irreducible 2-dimensional $GF(49)SL_2(5)$-module or the faithful irreducible 4-dimensional $GF(7)SL_2(5)$-module, and for $q = 9$ to one of two quasiequivalent faithful irreducible 4-dimensional $GF(7)SL_2(9)$-modules;*

(5) *$\overline{G}$ is isomorphic to one of the groups S_7, S_8, A_9, A_{10}, $PGL_2(49)$, $L_3(4) : 2_3$, $L_3(4).3.2_3$, $U_3(5)$, $U_3(5) : 2$, $U_3(5) : 3$, $U_3(5) : S_3$, $S_6(2)$, $O_8^+(2)$, $O_8^+(2) : 3$, or J_2;*

(6) *$\overline{G}$ is isomorphic to an extension of a nilpotent $\{3, 5\}$-group A of order*

divisible on 5 *by a group* B *such that* $F^*(B) = O_3(B) \times L$*, where the group* L *is isomorphic to* $L_2(7)$*, the group* $B/O_3(B)$ *is isomorphic to* $L_2(7)$ *or* $PGL_2(7)$*, any* 2*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module is isomorphic to one of two quasiequivalent irreducible* 3*-dimensional* $GF(3)L_2(7)$*-modules, and any* 5*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module is isomorphic either to the irreducible* 3*-dimensional* $GF(25)L_2(7)$*-module or to the absolutely irreducible* 6*-dimensional* $GF(5)L_2(7)$*-module;*

(7) $\overline{G}$ *is isomorphic to an extension of a nontrivial nilpotent* $\{2,5\}$*-group* A *by a group* B *such that* $F^*(B) = O_3(B) \times L$*, where the group* L *is isomorphic to* A_7 *or* $U_3(3)$*,* $|B : F^*(B)| \leq 2$*, and any* p*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module is isomorphic to the irreducible* 6*-dimensional* $GF(p)L$*-module for* $p \in \{2,5\}$*;*

(8) $\overline{G}$ *is isomorphic to an extension of a nontrivial nilpotent* $\{2,5\}$*-group* A *by a group* B *such that* $F^*(B) = O_3(B) \circ L$*, where* $L \cong 3^{\cdot}A_7$*, the group* $B/O_3(B)$ *is isomorphic to* L *or* $Aut(L)$*, any* p*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module for for* $p \in \{2,5\}$ *is isomorphic either to the faithful irreducible* 6*-dimensional* $GF(p^2)L$*-module or to the unfaithful irreducible* 6*-dimensional* $GF(p)L$*-module with the core of order* 3 *(see the item* (6)*);*

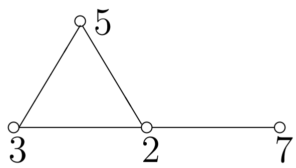
(9) $\overline{G}$ *is isomorphic to an extension of a nontrivial* 5*-group* A *by a group* B *such that* $F^*(B) = O_3(B) \circ L$*, where* $L \cong SU_3(5)$*, the group* $B/O_3(B)$ *is isomorphic to a subgroup from* $Aut(L)$*, any* 5*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module is isomorphic to the faithful irreducible* 3*-dimensional or* 6*-dimensional* $GF(25)L$*-module;*

(10) $\overline{G}$ *is isomorphic to an extension of a nontrivial* 2*-group* A *by a group* B *such that* $F^*(B) = O_3(B) \circ L$*, where the group* L *is isomorphic to* A_8*,* $S_6(2)$*,* $3^{\cdot}U_4(3)$ *or* J_2*, the group* $B/O_3(B)$ *is isomorphic to a subgroup from* S_8*,* $S_6(2)$*,* $U_4(3).2_{2/3}$ *or* J_2*, respectively, any* 2*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module is isomorphic to the faithful irreducible* 6*-dimensional* L*-module over the field* $GF(2)$ *for the first and second cases and over the field* $GF(4)$ *for the remaining cases;*

(11) $\overline{G}$ *is isomorphic to an extension of a nontrivial* 2*-group* A *by a group* B *such that* $F^*(B) = O_3(B) \circ L$*, where the group* L *is isomorphic to* $L_3(4)$ *or* $SL_3(4)$*, the group* $B/O_3(B)$ *is isomorphic to a subgroup from* $L_3(4).6$ *or* $L_3(4).3.2_3$*, respectively, any* 2*-chief factor of the group* $\overline{G}^\infty$ *as* L*-module is isomorphic either to the natural* 3*-dimensional* $GF(4)SL_3(4)$*-module or to one of two quasiequivalent unfaithful irreducible* 9*-dimensional* $GF(2)SL_3(4)$*-modules with the core of order* 3*.*

Each from items (1)–(11) *of the theorem is realised.*

Since the different prime graphs $\Gamma(Aut(J_2))$ and $\Gamma(A_{10})$ are isomorphic as abstract graphs, the arguments in the proofs of Theorems 8 and 9 are similar. It is interesting that $\Gamma(Aut(J_2)) = \Gamma(2 \times J_2)$ and $\Gamma(A_{10}) = \Gamma(3 \times J_2)$. The graph $\Gamma(Aut(J_2))$ has the form



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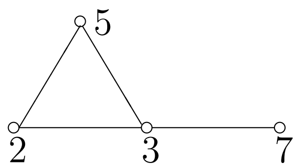
The graph $\Gamma(A_{10})$ has the form



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Recently AK (Sib. Electron. Mat. Izv., 2014) continued these investigations with the purpose of the study of finite 5-primary groups with disconnected prime graph. It was made a necessary preliminary step for this by the determining the finite almost simple 5-primary groups and their prime graphs. In addition, lists of finite simple 5-primary groups obtained by Jafarzadeh and Iranmanesh (London Math. Soc. Lecture Note Ser., 2007) and Zhang, Shi, Lv, Yu, Chen (to appear) are essentially refined. In particular, the following theorem is proved (it is provided in corrected and refined form).

Theorem 10. *A finite almost simple group G with the socle P is 5-primary if and only if one of the following statements holds:*

(1) *P is isomorphic to one of the groups A_{11}, A_{12}, $L_2(q)$ for $q \in \{2^6, 2^8, 2^9, 5^3, 5^4, 7^3, 7^4, 11^2, 17^2, 19^2\}$, $L_3(9)$, $L_3(27)$, $L_4(q)$ for $q \in \{4, 5, 7\}$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $U_3(q)$ for $q \in \{16, 17, 25, 81\}$, $U_4(q)$ for $q \in \{4, 5, 7, 9\}$, $U_5(3)$, $U_6(2)$, $S_4(q)$ for $q \in \{8, 16, 17, 25, 49\}$, $S_6(3)$, $S_8(2)$, $O_7(3)$, $O_8^+(3)$, $O_8^-(2)$, $G_2(q)$ for $q \in \{4, 5, 7, 8\}$, M_{22}, J_3, HS, He or M^cL;*

(2) *$G \cong L_2(2^p)$, where $p \geq 11$ is a prime and $|\pi(2^{2p} - 1)| = 4$;*

(3) *$G \cong Aut(L_2(2^p))$, where $p \geq 7$, $2^p - 1$ and $(2^p + 1)/3$ are some different primes;*

(4) *$G \cong Aut(L_2(3^p))$ or $O^2(Aut(L_2(3^p)))$, where $p \geq 5$ is a prime and $|\pi((3^p - 1)/2)| = |\pi((3^p + 1)/4)| = 1$;*

(5) *$G \cong L_2(p)$ or $PGL_2(p)$, where $p \geq 29$ is a prime and $|\pi(p^2 - 1)| = 4$;*

(6) *$G \cong L_2(p^r)$ or $PGL_2(p^r)$, where $p \in \{3, 5, 7, 17\}$, r is a prime, $3 < r \neq p$ and $|\pi(p^{2r} - 1)| = 4$;*

(7) *$U_3(2^p) \leq G \leq PGU_3(2^p) : 2$, where $p \geq 5$ and $2^p - 1$ are primes, $|\pi((2^p + 1)/3)| = |\pi((2^{2p} - 2^p + 1)/3)| = 1$;*

(8) *$P \cong L_3^\epsilon(p)$, where $\epsilon \in \{+, -\}$, p is a prime, $17 \neq p \geq 11$, $|\pi(p^2 - 1)| = 3$, and $|\pi(\frac{p^2 + \epsilon p + 1}{(3, p - \epsilon 1)})| = 1$;*

(9) *$G \cong S_4(p)$ or $PGSp_4(p)$, where $p \geq 11$ is a prime, $|\pi(p^2 - 1)| = 3$ and $p^2 + 1 = 2r$ or $2r^2$ for an odd prime r;*

(10) *$G \cong Sz(2^p)$, where $p \geq 7$ and $2^p - 1$ are primes and $|\pi(2^{2p} + 1)| = 3$;*

(11) *$G \cong Aut(Sz(8))$.*

Theorem 10 shows that finite simple 5-primary groups besides the groups $L_4(q)$ for $q \in \{4, 7\}$ and $U_4(q)$ for $q \in \{4, 5, 7, 9\}$ have disconnected prime graph.

Using Theorems 3 and 5, AK and his post-graduate Kolpakova (Zh. Fund. Prikl. Mat., 2015, to appear) described the chief factors of the commutator subgroups of finite nonsolvable groups G with disconnected Gruenberg-Kegel graph having exactly 5 vertices in the case when $G/F(G)$ is an almost simple n-primary group for $n \leq 4$.

Recently AK and Kolpakova (2015, to appear) determine the finite almost simple 6-primary groups and their prime graphs. By this, a list of finite simple 6-primary groups obtained by Jafarzadeh and Iranmanesh (London Math. Soc. Lecture Note Ser., 2007) is essentially refined.

Theorem 11. *A finite almost simple group G with the socle P is 6-primary if and only if one of the following statements holds:*

(1) *P is isomorphic to one of the groups*

A_n *for* $n \in \{13, 14, 15, 16\}$, $L_2(q)$ *for* $q \in \{2^{10}, 2^{16}, 3^6, 3^8, 3^{10}, 5^5, 11^4, 17^3, 17^4\}$, $L_3(q)$ *for* $q \in \{2^4, 2^7, 2^9, 5^2, 7^2\}$, $L_4(q)$ *for* $q \in \{2^3, 3^2, 17\}$, $L_5(7)$, $L_6(3)$, $L_7(2)$, $U_3(q)$ *for* $q \in \{2^9, 3^3, 5^3, 5^4, 7^2, 7^3, 17^2\}$, $U_4(q)$ *for* $q \in \{2^3, 2^4, 5^2\}$, $U_5(q)$ *for* $q \in \{4, 5, 9\}$, $U_6(3)$, $U_7(2)$, $O_7(q)$ *for* $q \in \{5, 7\}$, $O_9(3)$, $PSp_4(q)$ *for* $q \in \{2^5, 3^3, 3^4, 3^5, 11^2, 17^2\}$, $PSp_6(q)$ *for* $q \in \{4, 5, 7\}$, $PSp_8(3)$, $O_8^+(q)$ *for* $q \in \{4, 5, 7\}$, $O_8^-(3)$, $O_{10}^+(2)$, $O_{10}^-(2)$, ${}^3D_4(q)$ *for* $q \in \{4, 5\}$, $G_2(q)$ *for* $q \in \{3^2, 17\}$, ${}^2G_2(3^3)$, $F_4(2)$, Suz, Ru, Co_2, Co_3, M_{23}, M_{24}, J_1, Fi_{22}, HN*;*

(2) $G \cong Aut(L_2(2^r))$, *where* $r \geq 11$ *is a prime,* $r \notin \pi(P)$ *and* $|\pi(2^{2r} - 1)| = 4$*;*

(3) $G \cong L_2(2^r)$, *where* $r \geq 37$ *is a prime,* $r \notin \pi(P)$ *u* $|\pi(2^{2r} - 1)| = 5$*;*

(4) $G \cong L_2(2^{2r})$ *or* $O^r(Aut(L_2(2^{2r})))$, *where* $r \geq 7$ *u* $2^r - 1$ *are odd primes,* $r \notin \pi(G)$, $|\pi(\frac{2^r+1}{3})| = 1$ *and* $|\pi(2^{2r} + 1)| = 2$*;*

(5) $G \cong L_2(2^{r^2})$, *where* $r \geq 7$ *is a prime,* $r \notin \pi(G)$, $|\pi(2^{r^2} - 1)| = 2$ *and* $|\pi(2^{r^2} + 1)| = |\pi(2^{2r} - 1)| = 3$*;*

(6) $P \cong L_2(3^{2r})$ *and* $G \leq O^r(Aut(P))$, *where* $r > 13$ *is a prime,* $r \notin \pi(P)$, $|\pi(\frac{3^r-1}{2})| = |\pi(\frac{3^r+1}{4})| = 1$ *and* $|\pi(\frac{3^{2r}+1}{2})| = 2$*;*

(7) $P \cong L_2(3^{r^2})$ *or* $O^r(Aut(P))$, *where* r *is a prime,* $r \notin \pi(P)$, $|\pi(\frac{3^r+1}{4})| = |\pi(\frac{3^r-1}{2})| = 1$ *and* $|\pi(3^{2r^2} - 1)| = 5$*;*

(8) $P \cong L_2(p)$, *where* $p \geq 131$ *is a prime and* $|\pi(p^2 - 1)| = 5$*;*

(9) $P \cong L_2(p^2)$, *where* $p \geq 29$ *is a prime,* $|\pi(p^2-1)| = 4$ *and* $|\pi(p^2+1)| = 2$*;*

(10) $P \cong L_2(p^2)$, *where* $p \geq 13$ *is a prime and* $|\pi(p^2-1)| = |\pi(p^2+1)| = 3$*;*

(11) $G \cong L_2(p^r) : r$ *or* $Aut(L_2(p^r))$, *where* $p \in \{3, 5, 7, 17\}$ *and* r *are primes,* $r \notin \pi(P)$, $p^r \equiv \varepsilon 1 \pmod 4$ *for* $\varepsilon \in \{+, -\}$, $|\pi(p^r - \varepsilon 1)| = 2$ *and*

$|\pi(\frac{p^r+\varepsilon 1}{2})|=2$;

(12) $G\cong L_2(p^r)$ *or* $PGL_2(p^r)))$, *where* p *and* r *are odd primes*, $r\notin\pi(P)$ *and* $|\pi(p^{2r}-1)|=5$;

(13) $P\cong L_3^{\varepsilon}(2^r)$ *and* $G\leq O^r(Aut(P))$, *where* $\varepsilon\in\{+,-\}$, r *and* 2^r-1 *are primes*, $r\geq 5$ *for* $\varepsilon=+$ *and* $r\geq 19$ *for* $\varepsilon=-$, $r\notin\pi(P)$, $|\pi(\frac{2^r+1}{3})|=1$ *and* $|\pi(\frac{2^{2r}+\varepsilon 2^r+1}{(3,2^r-\varepsilon 1)})|=2$;

(14) $G\cong L_3^{\varepsilon}(3^r)$ *or* $L_3^{\varepsilon}(3^r):2$, *where* $\varepsilon\in\{+,-\}$, r *is a prime*, $r\geq 7$ *for* $\varepsilon=+$ *and* $r\geq 5$ *for* $\varepsilon=-$, $r\notin\pi(P)$, $|\pi(\frac{3^r-1}{2})|=|\pi(\frac{3^r+1}{4})|=1$ *and* $|\pi(3^{2r}+\varepsilon 3^r+1)|=2$;

(15) $P\cong L_3^{\varepsilon}(p)$, *where* $p\geq 41$ *is a prime*, $\varepsilon\in\{+,-\}$, $|\pi(p^2-1)|=4$ *u* $|\pi(\frac{p^2+\varepsilon p+1}{(3,p-\varepsilon 1)})|=1$;

(16) $P\cong L_3^{\varepsilon}(p)$, *where* p *is a prime*, $\varepsilon\in\{+,-\}$, $p\geq 11$ *for* $\varepsilon=+$ *and* $p\geq 31$ *for* $\varepsilon=-$, $|\pi(p^2-1)|=3$ *and* $|\pi(\frac{p^2+\varepsilon p+1}{(3,p-\varepsilon 1)})|=2$;

(17) $P\cong U_3(2^r)$ *and* $P:r\leq G$, *where* $r\geq 5$ *and* 2^r-1 *are primes*, $r\notin\pi(G)$ *and* $|\pi(\frac{2^r+1}{3})|=|\pi(\frac{2^{2r}-2^r+1}{3})|=1$;

(18) $P\cong L_4^{\varepsilon}(p)$, *where* $\varepsilon\in\{+,-\}$, p *is a prime*, $p\geq 19$ *for* $\varepsilon=+$ *and* $p\geq 11$ *for* $\varepsilon=-$, $|\pi(p^2-1)|=3$ *and* $|\pi(\frac{p^2+\varepsilon p+1}{(3,p-\varepsilon 1)})|=|\pi(\frac{p^2+1}{2})|=1$;

(19) $G\cong PSp_4(2^r)$, *where* $r>5$ *and* 2^r-1 *are primes*, $r\notin\pi(G)$, $|\pi(\frac{2^r+1}{3})|=1$ *and* $|\pi(2^{2r}+1)|=2$;

(20) $G\cong PSp_4(3^r)$ *или* $PGp_4(3^r)$, *where* $r>5$ *is a prime*, $r\notin\pi(G)$, $|\pi(\frac{3^r-1}{2})|=|\pi(\frac{3^r+1}{4})|=1$ *and* $|\pi(3^{2r}+1)|=3$;

(21) $P\cong PSp_4(p)$, *where* $p\geq 29$ *is a prime*, $|\pi(p^2-1)|=4$ *and* $|\pi(\frac{p^2+1}{2})|=1$;

(22) $P\cong PSp_4(p)$, *where* $p\geq 13$ *is a prime*, $|\pi(p^2-1)|=3$ *and* $|\pi(\frac{p^2+1}{2})|=2$;

(23) $G\cong G_2(p)$, *where* $p\geq 13$ *is a prime*, $|\pi(p^2-1)|=3$ *and* $|\pi(\frac{p^2+\epsilon p+1}{(3,p-\epsilon 1)})|=1$ *for* $\epsilon\in\{+,-\}$;

(24) $G\cong Sz(2^r)$, *where* $r\geq 13$ *is a prime*, $r\notin\pi(P)$, $|\pi(2^r-1)|=1$ *and* $|\pi(2^{2r}+1)|=4$;

(25) $G\cong Sz(2^r)$, *where* $r\geq 11$ *is a prime*, $r\notin\pi(P)$, $|\pi(2^r-1)|=2$ *and* $|\pi(2^{2r}+1)|=3$;

(26) $G\cong Aut(Sz(2^r))$, *where* $r\geq 7$ *and* 2^r-1 *are primes and* $|\pi(2^{2r}+1)|=3$.

By Jafarzadeh and Iranmanesh (London Math. Soc. Lecture Note Ser., 2007), the following Problem 3.12 was posed: *For which power primes q does $q^2 - 1$ have at most five different prime divisors?*

The case of this problem when $|\pi(q^2 - 1)| \leq 2$ is very known (see, for example, the paper by G. Higman (J. London Math. Soc., 1957) or the paper by Herzog (J. Algebra, 1968)): $|\pi(q^2 - 1)| \leq 2$ if and only if $q \in \{2, 3, 4, 5, 7, 8, 9, 17\}$.

The cases when the number $|\pi(q^2 - 1)|$ is equal to 3, 4, 5 considered by AK and Khramtsov (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2010), AK and Khramtsov (Trudy Trudy Inst. Mat. Mekh. UrO RAN, 2011), AK (Sib. Electron. Mat. Izv., 2014) and Kolpakova and AK (2015, to appear), respectively. Therefore, a classification of power primes q such that $|\pi(q^2 - 1)| \leq 5$ is obtained. A further refinement of the classification brings often to some Diophantine equations whose solving is difficult even for the modern number theory. For example, the question on the finiteness of the set of prime powers q such that $|\pi(q^2 - 1)| = 3$ is equivalent to the open Shi's Question 13.65 from "The Kourovka Notebook".

We consider an other very interesting for us Diophantine equation.

Nagell (1920) and Ljunggren (1943) studied equation of the form

$$(x^n - 1)/(x - 1) = y^m.$$

in integers $x, y, m > 1$, $n > 2$.

Later on the equation is investigated by many number theorists with using contemporary methods of Diophantine approximation. There exists the conjecture that the set of solutions (x, y, n, m) of this equation is finite and, it is possible, is exhausted by quadruples $(3, 11, 5, 2)$, $(7, 20, 4, 2)$, $(18, 7, 3, 3)$.

Most strong results in the solving this equation are obtained in works of Bugeaud, Mignotte, Roy, Shorey (Math. Proc. Cambridge Philos. Soc, 1999), Bugeaud и Mignotte (Pacific J. Math., 2000), Benett (J. reine Angew. Math., 2001): the equation has no solutions if x is a square; has unique solution $(x, y, n, m) = (3, 11, 5, 2)$ if $n \equiv 1 \ (mod\ m)$; if $(x, y, n, m) \neq (18, 7, 3, 3)$ and m is a prime then there exists a prime divisor p of x such that m divides $p - 1$.

Lucido (Boll. Unione Mat. Ital., 2002) described finite simple groups G such that the connected components of the graph $\Gamma(G)$ are trees, i. e. connected graphs without cycles. Furthermore, in this paper Lucido described the structure of a finite group whose the prime graph is a tree. My PhD student O.A. Alekseeva and myself consider more general problem of *the description of the structure of a finite group whose the prime graph contains no triangles (3-cycles).*

It is easy to see that if G is a finite group whose prime graph contains no triangles then its quotient $G/S(G)$ by the solvable radical $S(G)$ is almost simple.

In the case, when the investigated group is almost simple, we obtained the following result.

Theorem 12. *Let G be a finite almost simple group with the socle P. If the graph $\Gamma(G)$ contains no triangles then one of the following statements holds:*

(1) *P is isomorphic to one of the groups A_n для $n \in \{5,6,7\}$, $L_2(q)$ for $q \in \{7, 2^3, 3^4, 11, 13, 17, 5^2, 7^2, 2^9\}$, $L_3(q)$ for $q \in \{3,4,5,17\}$, $U_3(q)$ for $q \in \{3,7\}$, $L_4(3)$, $U_4(q)$ for $q \in \{2,3\}$, $G_2(3)$, ${}^2F_4(2)'$, M_{11}, M_{22};*

(2) *G is isomorphic to one of the groups A_8, $L_2(q)$ for $q \in \{2^4, 2^6\}$, $L_3(q)$ for $q \in \{7,8,9\}$, $L_3(7):2$, $L_3(9):2$, $U_3(q)$ for $q \in \{4,5,8\}$, $U_3(5):2$, $U_3(8):3$, $U_5(2)$, ${}^2G_2(27)$;*

(3) *$P \cong L_2(q)$ for $q \in \{5^3, 17^2\}$ and $PGL_2(q) \not\leq G$;*

(4) *$P \cong L_2(q)$, where $q \in \{2^p, 3^p\}$, p is an odd prime and $|\pi(q-1)| \leq 2 \geq |\pi(q+1)|$;*

(5) *$G \cong L_2(p)$, where $p > 17$ is a prime and $|\pi(p-1)| \leq 2 \geq |\pi(p+1)|$;*

(6) *$G \cong PGL_2(p)$, where $17 < p \neq 2^n \pm 1$ is a prime and $|\pi(p^2-1)| = 3$;*

(7) *$G \cong L_2(q)$, where $q = p^r$, $p \in \{3,5,7,17\}$, r is a prime, r does not divide $|G|$, $q \equiv \varepsilon 1$ (mod 4) for $\varepsilon \in \{+,-\}$, $|\pi(q-\varepsilon 1)| = \pi((q+\varepsilon 1)/2)| = 2$;*

(8) *$G \cong U_3(q)$, where $q = 2^p$, $p \geq 5$, $q-1$ and $(q+1)/3$ are primes, $|\pi((q^2-q+1)/3)| = 1$ and p does not divide $|G|$;*

(9) *$P \cong L_3^{\epsilon}(p)$, where $\epsilon \in \{+,-\}$, $11 \leq p \neq 2^n \pm 1$ is a prime, $(p-\epsilon 1)_3 = 3$, $|\pi(p^2-1)| = 3$ and $|\pi((p^2+\epsilon p+1)/3)| = 1$;*

(10) *$P \cong Sz(2^f)$, where either $f = 9$, or f is an odd prime and $max\{|\pi(q-1)|, |\pi(q-\sqrt{2q}+1)|, |\pi(q+\sqrt{2q}+1)|\} \leq 2$;*

(11) *$G \cong {}^2G_2(q)$, where $q = 3^p$, $p \geq 5$ is a prime, $|\pi((q-1)/2)| = |\pi((q+1)/4)| = 1$ and $|\pi(q-\sqrt{3q}+1)| \leq 2 \geq |\pi(q+\sqrt{3q}+1)|$.*

Theorem 12 implies

Corollary. *Let G be a finite almost simple group and the graph $\Gamma(G)$ contains no triangles. Then*

(1) *each connected component of the graph $\Gamma(G)$ is a tree;*

(2) *if G is simple then the graph $\Gamma(G)$ is disconnected;*

(3) $|\pi(G)| \leq 8$ *and* $|\pi(G)| = 8$ *for* $G \cong Aut(Sz(2^9))$.

Note that Theorem 12 refines essentially an obtained by Lucido list of finite simple groups G such that the connected components of the graph $\Gamma(G)$ are trees. The proof of Theorem 12 uses the above-mentioned results on finite almost simple n-primary groups with $n \leq 6$.

In the case, when the investigated group is solvable, we obtained the following result.

Theorem 13. *Let G be a finite solvable group and the graph $\Gamma(G)$ contains no triangles. Then the following statements hold:*

(1) *If the graph $\Gamma(G)$ is disconnected then G is a Frobenius group or a 2-Frobenius group and the graph $\Gamma(G)$ has exactly two connected components which are 1- or 2-chains.*

(2) *If the graph $\Gamma(G)$ is connected then the graph $\Gamma(G)$ is one of the following: n-chain for $1 \leq n \leq 4$, 4-cycle, 5-cycle.*

(3) *If the graph $\Gamma(G)$ is 2-chain then the Fitting length $l_F(G)$ of G can be arbitrarily large.*

(4) *If the graph $\Gamma(G)$ is 3-chain then $l_F(G) \leq 6$.*

(5) *If the graph $\Gamma(G)$ is 4-chain then $l_F(G) \leq 4$.*

(6) *If the graph $\Gamma(G)$ is 4-cycle then $l_F(G) \leq 5$.*

(7) *If the graph $\Gamma(G)$ is 5-cycle then $l_F(G) = 3$.*

Moreover, for any isomorphic type of the graph $\Gamma(G)$, besides of 2-chain and 5-cycle, the maximum of the number $l_F(G)$ is achieved on a group G such that the factor group $G/O(G)$ is isomorphic to the group $2{\cdot}S_4^-$.

The items (1) and (2) of Theorem 13 are proved also in the work by Gruber, Keller, Lewis, Naughton and Strasser (J. Algebra, to appear). But their proof uses the classification of prime graphs of finite solvable groups and some combinatorial results. Our proof is direct, very short and uses no deep combinatorial results.

There exists a solvable group G of shape $(7^2 \times 13^2) \leftthreetimes 2{\cdot}S_4^-$ for which the graph $\Gamma(G)$ is 4-chain and $l_F(G) = 4$.

There exists a solvable group G of shape $(5^2 \times 13) \leftthreetimes (2 \times F_{21})$, where F_{21} is the Frobenius group of order 21, for which the graph $\Gamma(G)$ is 5-cycle and $l_F(G) = 3$.

But in the cases (4) and (6), the corresponding examples do not constructed.

In the general case of nonsolvable groups, using Theorems 12 and 13, we obtained the following result.

Theorem 14. *If G is a finite non-solvable group and the graph $\Gamma(G)$ contains no triangles then $|\pi(G)| \leq 8$ and $|\pi(S(G))| \leq 3$.*

Moreover, for the proof of Theorem 14, we obtain a detail description of the structure of groups G from Theorem 14 in the case when $\pi(S(G))$ contains a number which does not divide the order of the group $G/S(G)$ (if $|\pi(S(G))| = 3$ then this condition is true).

The problem of *the realizability of an abstract finite graph as the prime graph of a finite group* is interesting also for us. There are not many works devoted to the problem. In unpublished Bachelor work of I.N. Zharkov (2008), who was a student of V.D. Mazurov, it was proved that a chain is realizable as the prime graph of a group if and only if its length is at most 4.

In the above-mentioned work by Gruber, Keller, Lewis, Naughton and Strasser, the graphs which may be realized as the prime graphs of finite solvable groups are precisely determined.

The analogous problem were considered by H.P. Tong-Viet (J. Algebra, 2013) for the graph $\Delta(G)$ whose vertex set is the set all primes dividing irreducible character degrees of a finite group G and two vertices p and q are adjacent if and only if the product pq divides some irreducible character degree of G.

Of course, in general, the problem has negative solution. For example, Gruenberg—Kegel theorem and the description of connected components of the prime graph for all finite simple non-abelian groups imply that the graph consisting of five pairwise non-adajcent vertices (5-coclique) is not realizable as the prime graph of a finite group. But in the paper of A.L. Gavrilyuk, I.V. Khramtsov, AK and N.V. Maslova (Sib. Electron. Mat. Izv., 2014) it was shown that any graph with at most five vertices, besides of 5-coclique, is realizable as the prime graph of a finite group.

Recently N. V. Maslova (2015) gave a solution of the mentioned problem for all complete bipartite graphs $K_{m,n}$, where $K_{m,n}$ is the graph with $m+n$ vertices whose vertices can be divided into two disjoint subsets U and V such that $|U| = m$, $|V| = n$ and vertices are adjacent if and only if they belong to different subsets. Shi proved the following theorem.

Theorem 15. *Let Γ be a complete bipartite graph $K_{m,n}$, where $m \leq n$. Then the following statements hold*:

(1) Γ *is realizable as the Gruenberg–Kegel graph of a group if and only if* $m+n \leq 6$ *and* $(m,n) \neq (3,3)$;

(2) *if* $m+n \leq 6$ *and* $(m,n) \neq (3,3), (1,5)$ *then there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$*;

(3) *if* $(m,n) = (1,5)$ *and Γ is realizable as the Gruenberg–Kegel graph of a group G then* $\pi(G) = \{2,3,7,13,19,37\}$, $O_2(G) \neq 1$ *and* $G/O_2(G) \cong {}^2G_2(27)$.