t-Walk-regular graphs, scheme graphs and 2-partially metric association schemes

Jack Koolen

This is based on joint work with M. Cámara, E.R. van Dam and J. Park, and with Zhi Qiao and Shao Fei Du

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Yekaterinburg, August, 2015
Outline

1. *t*-Walk-regular graphs
   - Definitions
   - Examples
   - Partially distance-regular graphs

2. Some results
   - Adjacency algebra
   - Terwilliger

3. Examples with relatively many eigenvalues
   - A result of C. Dalfó et al.
   - Graphs from group divisible designs

4. Association schemes
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   - Multiplicity 3
   - Problems
Let $\Gamma = (V, E)$ be a graph.

The distance $d(x, y)$ between two vertices $x$ and $y$ is the length of a shortest path connecting them.

The maximum distance between two vertices in $\Gamma$ is the diameter $D = D(\Gamma)$.

The valency of $x$ is the number of vertices adjacent to it.

A graph is regular with valency $k$ if each vertex has $k$ neighbors.
Definitions

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- The distance $d(x, y)$ between two vertices $x$ and $y$ is the length of a shortest path connecting them.
- The maximum distance between two vertices in $\Gamma$ is the diameter $D = D(\Gamma)$.
- The valency of $x$ is the number of vertices adjacent to it.
- A graph is regular with valency $k$ if each vertex has $k$ neighbors.
- The adjacency matrix $A$ of $\Gamma$ is the matrix whose rows and columns are indexed by the vertices of $\Gamma$ and the $(x, y)$-entry is 1 whenever $x$ and $y$ are adjacent and 0 otherwise.
- The eigenvalues of the graph $\Gamma$ are the eigenvalues of $A$. 
A graph $\Gamma$ is called $t$-walk-regular if the number of walks of length $\ell$ between vertices $x$ and $y$ only depends on the distance between $x$ and $y$ and $\ell$, provided that such a distance does not exceed $t$. 
A graph $\Gamma$ is called \textit{t-walk-regular} if the number of walks of length $\ell$ between vertices $x$ and $y$ only depends on the distance between $x$ and $y$ and $\ell$, provided that such a distance does not exceed $t$.

\textit{t-Walk-regular} graphs are generalizations of distance-regular graphs. Many results on distance-regular graphs can be extended to the class of 2-walk-regular graphs, especially those results that uses Euclidean representations.
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Examples 1

There are many examples of $m$-walk-regular graphs that are not distance-regular.

- The bipartite double of the dodecahedron is 3-walk-regular but not 4-walk-regular. (Bipartite double: For every vertex $x$ create two vertices $x^+$ and $x^-$ and if $x \sim y$ then $x^\epsilon \sim y^\delta$ if $\epsilon \delta = -$.)

For example all 1-arc-transitive cubic graphs are 2-walk-regular. Any cubic graph is at most 5-arc-transitive (Tutte) and there are infinitely many connected non-isomorphic cubic 5-arc-transitive graphs.

Any $k$-regular graph is at most 7-arc-transitive (Weiss) and there are infinitely many connected non-isomorphic 7-arc-transitive 4-regular graphs (Conder and Walker(1998)).
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- $m$-Arc transitive graphs are at least $m$-walk-regular. ($m$-Arc-transitive graphs have an automorphism group transitive on the $m$-arcs, i.e. $(m+1)$-tuples $(x_0, x_1, \ldots, x_m)$ such that $x_i \sim x_{i+1}$ and $x_{i-1} \neq x_{i+1}$)
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- Any $k$-regular graph is at most 7-arc-transitive (Weiss) and there are infinitely many connected non-isomorphic 7-arc-transitive 4-regular graphs (Conder and Walker(1998))
Examples 2

Two generalizations of $m$-arc-transitive graphs:

- Partially $m$-distance-transitive graphs: Connected graph with diameter at least $m$ such that for any quadruple of vertices $x_1, x_2, y_1, y_2$ with $d(x_1, x_2) = d(y_1, y_2) \leq m$ there is an automorphism $\tau$ such that $x_i^\tau = y_i$ ($i = 1, 2$).
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- Praeger et al. (2010) also introduced the notion of $m$-geodetically-transitive graphs, i.e. the automorphism group is transitive on the $(m+1)$-tupels $(x_0, x_1, \ldots, x_m)$ with $x_i \sim x_{i+1}$ and $d(x_0, x_m) = m$. 
Question:

Are there partially $m$-distance-transitive graphs $\Gamma$ which are not $(m + 1)$-distance-transitive with $m < \text{diam}(\Gamma)$ with $m$ large?
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Are there partially $m$-distance-transitive graphs $\Gamma$ which are not $(m + 1)$-distance-transitive with $m < \text{diam}(\Gamma)$ with $m$ large? The same question for $m$-geodetically-transitive graphs.
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Let $\Gamma_i(x) := \{y \in V(\Gamma) \mid d(x, y) = i\}$.

We say $\Gamma$ is $t$-partially distance-regular ($t \leq D$) (with partial intersection array $\iota = \{b_0, \ldots, b_t; c_1 = 1, c_2, \ldots, c_t\}$) if $\#\Gamma_{i-1}(y) \cap \Gamma_1(x) = c_i$ and $\#\Gamma_{i+1}(y) \cap \Gamma_1(x) = b_i$ for $d(x, y) = i \leq t$ with the understanding that $b_D = 0$. 
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- If $t = D$, the graph is called distance-regular.
A distance-regular graph with diameter $D$ is $D$-walk-regular (Rowlinson).
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The last condition is much weaker than the first. Example: Take the folded $n$-cube $\tilde{Q}(n)$, i.e. you take the $n$-cube and you identify the antipodes. Take the cartesian product $K_2 \times \tilde{Q}(n)$. The resulting graph is about $n/2$-partially distance-regular but not even 1-walk-regular.
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Adjacency algebra

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First we need to look at the adjacency algebra for an \( m \)-walk-regular graph.

- \( \Gamma \) a graph with adjacency matrix \( A \).
- The adjacency algebra \( A \) is the matrix algebra generated by \( A \), i.e. the algebra consisting of all polynomials in \( A \) with coefficients in the real field.
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- Assume that \( \Gamma \) has distinct eigenvalues \( \theta_0 > \theta_1 > \cdots > \theta_d \).
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- $\Gamma$ a graph with adjacency matrix $A$.
- The adjacency algebra $\mathcal{A}$ is the matrix algebra generated by $A$, i.e. the algebra consisting of all polynomials in $A$ with coefficients in the real field.
- Assume that $\Gamma$ has distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$.
- Then $\dim(\mathcal{A}) = d + 1$ and $\mathcal{A}$ has primitive idempotents $E_i$, $i = 0, 1, \ldots, d$ such that $AE_i = \theta_i E_i$. 
Adjacency algebra 2

- Let $\Gamma$ be a connected graph, say with diameter $D$. 
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Let $A_i$ be the distance-$i$ matrix, i.e. $(A_i)_{xy} = 1$ if $d(x,y) = i$ and 0 otherwise.

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Let $A_i$ be the distance-$i$ matrix, i.e. $(A_i)_{xy} = 1$ if $d(x, y) = i$ and 0 otherwise.

Let $m \leq D$.

Then $\Gamma$ is $m$-walk-regular if and only if $A_i \circ E_j = c_{ij}A_i$ for some scalar $c_{ij}$ for all $0 \leq i \leq m$ and $0 \leq j \leq d$. 

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Note a 0-walk-regular graph is regular say with valency $k(=b_0)$. 
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Now I will give some results of Terwilliger that can be generalised to 2-walk-regular graphs.
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**Theorem**

Let $\Gamma$ be a connected 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Let $x$ be a vertex of $\Gamma$ and let $\Delta(x)$ has eigenvalues $a_1 = \eta_1 \geq \eta_2 \geq \cdots \geq \eta_k$. Then $b^- := -1 - \frac{b_1}{1+\theta_1} \leq \eta_k \leq \eta_2 \leq b^+ := -1 - \frac{b_1}{1+\theta_d}$. 
If one of the multiplicities is small we can say more.
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**Theorem**

Let $\Gamma$ be a connected coconnected (i.e. its complement is connected as well) 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ with respective multiplicities $m_0 = 1, m_1, \ldots, m_d$. If $m_i < k$ for $1 \leq i \leq d$ then

- $i = 1$ or $i = d$. 

$-b_i + 2m_i$ is an eigenvalue of $\Delta(x)$ with multiplicity at least $k - m_i$. 

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$k \leq (m_i + 2)(m_i - 1)/2$. 


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C. Dalfó et al. (2011) showed the following result.

**Proposition**

Let $s, d$ be positive integers. Let $\Gamma$ be a connected $s$-walk-regular graph with diameter $D \geq s$ and with exactly $d + 1$ distinct eigenvalues. Then the following hold:

- If $d \leq s + 1$, then $\Gamma$ is distance-regular;
- If $d \leq s + 2$ and $\Gamma$ is bipartite, then $\Gamma$ is distance-regular.
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Later we will construct infinitely many bipartite 2-walk-regular graphs with 6 eigenvalues, which are not distance-regular. So this shows that we can not do better for $s = 2$ in the second item.
Let us first find a 2-walk-regular graph with 5 distinct eigenvalues.
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Consider the line graph $\Lambda$ of $O_4$.

Then it easy to see that $\Lambda$ is 2-partially distance-transitive, so 2-walk-regular.
• Let us first find a 2-walk-regular graph with 5 distinct eigenvalues.
• Let $O_4$ be the Odd graph with valency 4.
• It has 35 vertices and distinct eigenvalues 4, 2, $-1$, $-3$.
• Consider the line graph $\Lambda$ of $O_4$.
• Then it easy to see that $\Lambda$ is 2-partially distance-transitive, so 2-walk-regular.
• It is easy to calculate that $\Lambda$ has exactly 5 distinct eigenvalues 6, 4, 1, $-1$ and $-2$. 
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It is an open problem, whether there exist infinitely many 2-walk-regular graphs with exactly 5 distinct eigenvalues, which are not distance-regular.
One way to construct them is to construct non-bipartite distance-regular graphs with diameter 3 and girth 6, and then take its line graph.
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Classical examples

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Let $X$ be the set of non-zero elements of $V$.

For $x \in X$, let $G_x = \{\alpha x \mid \alpha \in \text{GF}^*(q) := \text{GF}(q) \setminus \{0\}\}$ and $G := \{G_x \mid x \in X\}$. 

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Let $B := \{ x + H \mid x \in X, H \text{ a hyperplane in } V, x \notin H \}$, where $x + H = \{ x + h \mid h \in H \}$, the set of affine hyperplanes.
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Take two distinct elements in \( X \). If they are linearly dependent then there is no proper affine hyperplane they lie together in.

If they are linearly independent then there are exactly \( q^{r-2} \) proper affine hyperplanes they lie together in.
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- Let $B := \{x + H \mid x \in X, H \text{ a hyperplane in } V, x \not\in H\}$, where $x + H = \{x + h \mid h \in H\}$, the set of affine hyperplanes.
- Take two distinct elements in $X$. If they are linearly dependent then there is no proper affine hyperplane they lie together in.
- If they are linearly independent then there are exactly $q^{r-2}$ proper affine hyperplanes they lie together in.
- This shows:
The design \( D(r, q) := (X, G, B) \) is a group divisible design with the dual property with parameters \((q - 1, \frac{q^r - 1}{q - 1}; q^{r-1}; 0, q^{r-2})\), or in other words a \( \text{GDDDP}(q - 1, \frac{q^r - 1}{q - 1}; q^{r-1}; 0, q^{r-2}) \). (The dual property means that we can interchange the role of points and blocks to obtain a design with the same parameters).

It is clear that the general linear group \( \text{GL}(r, q) \) acts as a group of automorphisms of \( D(r, q) \) such that its subgroup \( Z := \{ \alpha I_r \mid \alpha \in \text{GF}^*(q) \} \) fixes the set \( G_x \) for all \( x \in X \).
The design $\mathcal{D}(r, q) := (X, \mathcal{G}, \mathcal{B})$ is a group divisible design with the dual property with parameters $(q - 1, \frac{q^r - 1}{q-1}; q^r - 1; 0, q^r - 2)$, or in other words a GDDDP$(q - 1, \frac{q^r - 1}{q-1}; q^r - 1; 0, q^r - 2)$. (The dual property means that we can interchange the role of points and blocks to obtain a design with the same parameters).

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Let $a$ be a primitive element of $\text{GF}^*(q^r)$.

Observe that $(\text{GF}^*(q^r), \cdot) = \langle a \rangle$ is a cyclic group of order $q^r - 1$. 
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Now define the map $\tau_a \in \text{GL}(r, q)$ by $\tau_a(x) = ax$ for $x \in \text{GF}^*(q^r)$. 

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• Let $a$ be a primitive element of $GF^*(q^r)$.

• Observe that $(GF^*(q^r), \cdot) = \langle a \rangle$ is a cyclic group of order $q^r - 1$.

• As we can consider $GF(q^r)$ as a vector space of dimension $r$ over $GF(q)$, with basis $\{a^i \mid i = 0, 1, 2, \ldots, r-1\}$.

• Now define the map $\tau_a \in GL(r, q)$ by $\tau_a(x) = ax$ for $x \in GF^*(q^r)$.

• Then $\tau_a$ generates a cyclic subgroup $C$ of order $q^r - 1$ in $GL(r, q)$.

• This shows that there is a cyclic group (the Singer group) of automorphisms that acts regularly on the points of the design. We will need this later.
Now we are going to construct a graph $\Gamma(r, q)$ from the design $D(r, q) := (X, G, B)$. 
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$x \in X$ is adjacent to $B \in B$ if $x$ lies in $B$.

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You can construct other GDDDP from these examples by considering certain subgroups of $C$.

It can be shown that the graphs $\Gamma(r, q)$ are 2-arc-transitive dihedrants, using the Singer group.

Du et al. classified the 2-arc-transitive dihedrants, but in their classification they did not have the graphs $\Gamma(r, q)$ with $q$ even.
Some 2-arc transitive graphs

- Now we are going to construct a graph $\Gamma(r, q)$ from the design $\mathcal{D}(r, q) := (X, \mathcal{G}, \mathcal{B})$.
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- It is not so difficult to see that $\Gamma(r, q)$ has exactly 6 distinct eigenvalues and is 2-arc-transitive, so, in particular, it is 2-walk-regular.
- You can construct other GDDDP from these examples by considering certain subgroups of $C$.
- It can be shown that the graphs $\Gamma(r, q)$ are 2-arc-transitive dihedrants, using the Singer group.
- Du et al. classified the 2-arc-transitive dihedrants, but in their classification they did not have the graphs $\Gamma(r, q)$ with $q$ even.
- $\mathcal{D}(r, q)$ can also be constructed using relative difference sets. That is how we found the examples of Bose.
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   - Partially distance-regular graphs

2. Some results
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   - Terwilliger

3. Examples with relatively many eigenvalues
   - A result of C. Dalfó et al.
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4. Association schemes
   - Definitions
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5. Multiplicity results
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Let $X$ be a finite set with $n$ elements. A association scheme is a pair $(X, \mathcal{R})$ such that

1. $\mathcal{R} = \{R_0, R_1, \cdots, R_d\}$ is a partition of $X \times X$,
2. $R_0 = \Delta := \{(x, x) \mid x \in X\}$,
3. for each $i$ ($0 \leq i \leq d$) there exists $j$ such that $R_i = R_j^T$, i.e., if $(x, y) \in R_i$ then $(y, x) \in R_j$,
4. there are numbers $p_{ij}^h$ (the intersection numbers of $(X, \mathcal{R})$) such that for any pair $(x, y) \in R_h$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals $p_{ij}^h$. 

The elements $R_i$ are called the relations of $(X, \mathcal{R})$ and the number $d + 1$ of relations is called the rank of $(X, \mathcal{R})$.
Definitions 1

Let \( X \) be a finite set with \( n \) elements. A association scheme is a pair \((X, R)\) such that

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4. there are numbers \( p_{ij}^h \) (the intersection numbers of \((X, R)\)) such that for any pair \((x, y) \in R_h\) the number of \( z \in X \) with \((x, z) \in R_i\) and \((z, y) \in R_j\) equals \( p_{ij}^h \).

- The elements \( R_i \) are called the relations of \((X, R)\) and the number \( d + 1 \) of relations is called the rank of \((X, R)\).
- If \( R_i^T = R_i \), then we call the relation \( R_i \) symmetric.
- If all relations are symmetric, we call the scheme symmetric.
Definitions 2

Let $A_i$ be the relation matrix with respect to $R_i$ such that the rows and the columns of $A_i$ are indexed by the elements of $X$ and the $(x, y)$-entry is 1 whenever $(x, y) \in R_i$ and 0 otherwise.
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Then the conditions (i)-(iv) are expressed by:

$(i)' \quad \sum_{i=0}^{d} A_i = J$, where $J$ is the all-one matrix,

$(ii)' \quad A_0 = I$, where $I$ is the identity matrix,

$(iii)' \quad$ For all $i$ there exists $j$ such that $(A_i)^T = A_j$,

$(iv)' \quad A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$.

The Bose-Mesner Algebra $\mathcal{M}$ is the matrix algebra generated by the relation matrices (over $\mathbb{C}$).

$\mathcal{M}$ has a basis of primitive idempotents called scheme idempotents if the scheme is symmetric.
An association scheme \((X, R)\) with rank \(d + 1\) is called \(t\)-partially metric (with respect to a symmetric relation \(R\)) if there exists an ordering of the relation matrices \(A_0 = I, A_1, \ldots, A_d\) such that \(A_i\) is a polynomial of degree \(i\) in \(A\) for \(i = 1, 2, \ldots, t\), where \(A\) is the relation matrix of \(R\).
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Note that \(A_1 = A\) as the relation matrices are \((0, 1)\)-matrices and that we assume that if \((X, R)\) is \(t\)-partially metric, then we always assume to have this ordering of the relation matrices \(A_0 = I, A_1, \ldots, A_d\) such that \(A_i\) is a polynomial of degree \(i\) in \(A\) for \(i = 1, 2, \ldots, t\).
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- A (symmetric) association scheme with rank \(d + 1\) is called metric if it is \(d\)-partially metric.
We are going to construct graphs from association schemes.

- A graph $\Gamma$ is called the **scheme graph** of $(X, \mathcal{R})$ (with respect to $R$) if the adjacency matrix $A$ of $\Gamma$ is equal to the relation matrix of $R$. In this case, we call the relation $R$ the **corresponding relation** of $\Gamma$.

- We call the relation $R$ **connected** if the corresponding scheme graph is connected.
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- If relation $R$ is the corresponding relation for a $t$-partially metric scheme, then the corresponding scheme graph is $t$-walk-regular.
We are going to construct graphs from association schemes.

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- If relation $R$ is the corresponding relation for a $t$-partially metric scheme, then the corresponding scheme graph is $t$-walk-regular.

- If the scheme is metric then the corresponding scheme graph is **distance-regular**.
The bipartite double of an association scheme \((X, R_0, R_1, \ldots, R_d)\) is the scheme \((X \times \{+, -\}, R_0^+, R_0^-, \ldots, R_d^+, R_d^-)\), where \((x, \epsilon)\) and \((y, \delta)\) are in relation \(R_i^{\epsilon\delta}\) when \(x, y\) are in relation \(R_i\).
Bipartite double

- The bipartite double of an association scheme \((X, R_0, R_1, \ldots, R_d)\) is the scheme \((X \times \{+, -\}, R_0^+, R_0^-, \ldots, R_d^+, R_d^-)\), where \((x, \epsilon)\) and \((y, \delta)\) are in relation \(R_i^{\epsilon \delta}\) when \(x, y\) are in relation \(R_i\).

- If \(\Gamma\) is the scheme graph \(\Gamma\) of relation \(R_i\) in the original scheme, then the scheme graph of relation \(R_i^-\) is the bipartite double of \(\Gamma\).
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   - Examples
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2. Some results
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3. Examples with relatively many eigenvalues
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   - Definitions
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   - Multiplicity 3
   - Problems
Most of the 2-partially metric association schemes come from groups. I will describe the scheme graphs of some examples.

- The $t$-arc-transitive graphs are scheme graphs of $t$-partially metric association schemes, but those schemes are usually not symmetric. These graphs have $c_2 = 1$ if $t \geq 2$. 
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- The $t$-arc-transitive graphs are scheme graphs of $t$-partially metric association schemes, but those schemes are usually not symmetric. These graphs have $c_2 = 1$ if $t \geq 2$.

- The bipartite double of the dodecahedron is the scheme graph of two different symmetric association schemes, namely the bipartite double scheme $BD$ of the metric scheme of the dodecahedron and a fusion scheme of $BD$. The scheme $BD$ is 2-partially metric, whereas the latter scheme is 3-partially metric.
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- The symmetric bilinear forms graphs $SBF(n, q)$ have as vertices the $n \times n$ symmetric matrices over a finite field $GF(q)$ (where $q$ is a prime power) and two matrices are adjacent if their difference has rank 1. These graphs have $c_2 \geq 2$ and are locally the disjoint union of cliques. For $n \geq 4$ they are 2-distance-transitive but not distance-regular.
Examples 2

- De Caen et al. found an infinite family of triangle-free distance-regular antipodal graphs of diameter 3. If you take the bipartite double of these graphs you obtain 2-walk-regular graphs with $c_2 = 2$ and $a_1 = 0$. They are also the scheme graphs of the bipartite double scheme of the underlying metric scheme, and this scheme is 2-partially metric and symmetric.
Examples from codes

- Let $C$ be a binary linear code, say of length $n$, i.e. a subspace of the $n$-dim space $GF(2)^n$.
- Let $\Gamma(C)$ be the coset graph of $C$, i.e. the vertices are the cosets $x + C$ of $C$ and two cosets are adjacent if there is an edge between them in the Hamming graph.
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- If the minimum distance in $C$ is at least $2t \geq 2$, then $c_i = i$ and $a_i = 0$ for $i \leq t$.
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- If the minimum distance in $C$ is at least $2t \geq 2$, then $c_i = i$ and $a_i = 0$ for $i \leq t$.
- But usually the coset graph $\Gamma(C)$ is not 2-walk-regular.
- If the automorphism group of the code $C$ acts 2-transitive on the positions and the minimum weight is at 4, then $\Gamma(C)$ is partially 2-distance-transitive.
Examples from codes 2

- Let $C$ be the truncated code of the even sub code of the Golay code. Then $\Gamma(C)$ is distance-transitive.
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- Let $C$ be the truncated code of the even sub code of the Golay code. Then $\Gamma(C)$ is distance-transitive.
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There are some more examples which can be constructed from certain sub codes of the Golay, but those that are 3-distance-transitive are also distance-transitive.
Examples from codes 2

- Let $C$ be the truncated code of the even sub code of the Golay code. Then $\Gamma(C)$ is distance-transitive.
- The bipartite double of $\Gamma(C)$ is 3-distance-transitive, and is the coset graph of the even sub code of $C$.
- There are some more examples which can be constructed from certain sub codes of the Golay, but those that are 3-distance-transitive are also distance-transitive.
- Let $C$ be the simplex $t$-dimensional code over the binary field, i.e. the dual code of a Hamming code of length $2^t - 1$. Then the coset graph $\Gamma(C)$ is 2-distance-transitive but not 3-walk-regular.
- There are many more examples of coset graphs that are 2-distance-transitive. We are still working in this.
Examples from designs

- The graph $\Gamma(r, q)$ constructed from the group divisible designs above comes from a five-class association scheme.
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- The graph $\Gamma(r, q)$ constructed from the group divisible designs above comes from a five-class association scheme.
- Let $\mathcal{D}$ be a 2-design for which the automorphism group of the design acts 2-transitive on the points.
- Let $C$ be the linear code generated by the support of the blocks. Note it makes a difference here whether you look at the design or at its complementary design, i.e. the blocks are the complements of the blocks of the original design.
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- The graph $\Gamma(r, q)$ constructed from the group divisible designs above comes from a five-class association scheme.
- Let $D$ be a 2-design for which the automorphism group of the design acts 2-transitive on the points.
- Let $C$ be the linear code generated by the support of the blocks. Note it makes a difference here whether you look at the design or at its complementary design, i.e. the blocks are the complements of the blocks of the original design.
- For example take as your design the projective plane of order a power of 2. (For odd order you obtain the trivial code $GF(2)^n$.) The dimension of this code has been determined long ago by many people. We can show the coset graph is 2-distance-transitive. We are still trying to determine whether they are 3-walk-regular.
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Theorem 1 [2013,CDKP]

Let $\Gamma$ be a 2-walk-regular graph, different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$ with multiplicity 3.
2-Walk-regular graphs with multiplicity 3

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Let $\Gamma$ be a 2-walk-regular graph, different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$ with multiplicity 3. Then $\Gamma$ is a cubic graph with $a_1 = a_2 = 0$ (i.e. there are no triangles nor pentagons), the dodecahedron, or the icosahedron.
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Cubic 2-walk-regular graphs?????
An family of cubic 2-walk-regular graphs with multiplicity 3

In 2002, Feng and Kwak constructed a family of arc-transitive covers of the cube as voltage graphs and this family gives an infinite family of cubic 2-walk-regular graphs with eigenvalue $\pm 1$ with multiplicity 3.
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But if you assume that the graph comes from a symmetric association scheme with a connecting relation with valency 3, then we can classify them.
Our Result:

**Theorem 2**

Let $(X, \mathcal{R})$ be a 2-partially metric association scheme with corresponding relation $R$, corresponding valency $k \geq 3$ and rank $d + 1 \geq 3$. Let $E_0, E_1, \ldots, E_d$ be the minimal scheme idempotents with corresponding eigenvalues $\theta_0 = k, \theta_1, \ldots, \theta_d$ and multiplicities $m_0 = 1, m_1, \ldots, m_d$ respectively. Let $\Gamma$ be the scheme graph of $(X, \mathcal{R})$. If there exists an integer $i$ ($1 \leq i \leq d$) such that $m_i = 3$, then one of the following holds:

(i) $\Gamma$ is the cube,

(ii) $\Gamma$ is the Möbius-Kantor graph (a 2-cover of the cube),

(iii) $\Gamma$ is the Nauru graph (a 3-cover of the cube),

(iv) $\Gamma$ is the dodecahedron,

(v) $\Gamma$ is the bipartite double of the dodecahedron,

(vi) $\Gamma$ is the icosahedron,

(vii) $\Gamma$ is the octahedron,

(viii) $\Gamma$ is a regular complete 4-partite graph.

Moreover, the association scheme is uniquely determined by $\Gamma$. 
The bipartite double of the dodecahedron has no eigenvalue with multiplicity 3. As I remarked earlier, there are two symmetric 2-partially metric association schemes with this graph as its scheme graphs, namely the bipartite double scheme of the dodecahedron and a fusion scheme of this scheme.
Remarks

- The bipartite double of the dodecahedron has no eigenvalue with multiplicity 3. As I remarked earlier, there are two symmetric 2-partially metric association schemes with this graph as its scheme graphs, namely the bipartite double scheme of the dodecahedron and a fusion scheme of this scheme.

- The first scheme has four minimal idempotents with multiplicity 3, two of them have corresponding eigenvalue $\sqrt{5}$ and two of them have corresponding eigenvalue $-\sqrt{5}$. 
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- If the valency $k$ equals three, then we do not need to assume that the scheme is 2-partially metric, as that is implied by a result of N. Yamazaki (1998).
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The theorem is not true for symmetric association schemes (which are not 2-partially metric), as the $t$-coclique extensions of the dodecahedron show. (In this case the smallest non-trivial valency is equal to $3t$)
Ei. Bannai and Et. Bannai (2006) showed the following result:

**Theorem 3**

The scheme graph of a **primitive** (i.e. all non-trivial relations are connected) association scheme with a multiplicity 3 is the tetrahedron.
Ei. Bannai and Et. Bannai (2006) showed the following result:

**Theorem 3**
The scheme graph of a **primitive** (i.e. all non-trivial relations are connected) association scheme with a multiplicity 3 is the tetrahedron.

N. Yamazaki (1998) studied the symmetric association schemes with a relation with valency three. He showed:

**Theorem 4**
Let \((X, R_0, \ldots, R_d)\) be a symmetric association scheme with a connecting relation \(R\) of valency 3. Then the association scheme is metric with respect to \(R\) (and its corresponding scheme graph is distance-regular), or the corresponding scheme graph is bipartite.
Outline

1. t-Walk-regular graphs
   - Definitions
   - Examples
   - Partially distance-regular graphs

2. Some results
   - Adjacency algebra
   - Terwilliger

3. Examples with relatively many eigenvalues
   - A result of C. Dalfó et al.
   - Graphs from group divisible designs

4. Association schemes
   - Definitions
   - Examples

5. Multiplicity results
   - Multiplicity 3
   - Problems
We conclude this talk with some open problems.

- Are there only finitely many symmetric association schemes with a connecting relation with valency 3?

- For fixed $k \geq 3$, is 2-partially metric enough to show that there only finitely many symmetric association schemes with a connecting relation with valency $k$?
We conclude this talk with some open problems.

- Are there only finitely many symmetric association schemes with a connecting relation with valency 3?
- For fixed \( k \geq 3 \), is 2-partially metric enough to show that there only finitely many symmetric association schemes with a connecting relation with valency \( k \)?
- Find more examples of 3-partially metric symmetric schemes, which are not metric. On this moment we only know of about 3 or 4 examples.
- Find 2-walk-regular graphs which are locally connected. On this moment we do not have any example which is not distance-regular.
Thank you for your attention.