# ON THE PRONORMALITY OF SUBGROUPS OF ODD INDICES IN FINITE SIMPLE GROUPS

Natalia V. Maslova

IMM UB RAS, UrFU

Joint work with Anatoly Kondrat'ev and Danila Revin

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# DEFINITIONS

AGREEMENT. If another isn't established, we consider finite groups only.

Let $G$ be a group.

DEFINITION. A subgroup $H$ of a group $G$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

REMARK. $H$ is a normal subgroup of $G \Rightarrow H$ is pronormal in $G$.

QUESTION. What are pronormal subgroups of $G$?

EXAMPLES. The following subgroups are pronormal in finite groups:

- Normal subgroups;
- Maximal subgroups;
- Sylow subgroups.

PROPOSITION 1 (Evgeny Vdovin and Danila Revin, 2012). Let $G$ be a group, $H \leq G$ and $S \leq H$ for some pronormal (possibly, Sylow) subgroup $S$ of $G$. Then the following conditions are equivalent:

(1) $H$ is pronormal in $G$;
(2) subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

PROOF. Prove (2) $\Rightarrow$ (1). Let $g \in G$ and note $S, S^g \in \langle H, H^g \rangle$. $S$ in pronormal, so, there exists $y \in \langle S, S^g \rangle$ such that $S = S^{gy}$. In particular, $gy \in N_G(S)$. In view of (2) the subgroups $H$ and $H^{gy}$ are conjugate in $\langle H, H^{gy} \rangle$.

Let $t \in \langle H, H^{gy} \rangle$ and $H^t = H^{gy}$. Then $t^{-1}Ht = y^{-1}g^{-1}Hgy$ and $(yt^{-1}y^{-1})yHy^{-1}(yty^{-1}) = g^{-1}Hg$. Moreover, it's easy to see, $yty^{-1} \in \langle H^{y^{-1}}, H^g \rangle$. Thus, the subgroups $H^{y^{-1}}$ and $H^g$ are conjugate in $\langle H^{y^{-1}}, H^g \rangle$.

PROPOSITION 1 (Evgeny Vdovin and Danila Revin, 2012). Let $G$ be a group, $H \leq G$ and $S \leq H$ for some pronormal (possibly, Sylow) subgroup $S$ of $G$. Then the following conditions are equivalent:

(1) $H$ is pronormal in $G$;
(2) subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

PROOF. Recall, $y \in \langle S, S^g \rangle$ such that $S = S^{gy}$. We have proved, the subgroups $H^{y^{-1}}$ and $H^g$ are conjugate in $\langle H^{y^{-1}}, H^g \rangle$. The following Lemma completes the proof.

LEMMA (EXERCISE). Let $H$ be a subgroup of $G$, $g \in G$ and $y \in \langle H, H^g \rangle$. If the subgroups $H^y$ and $H^g$ are conjugate in $\langle H^y, H^g \rangle$ then the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$.

REMARK. Let $G$ be a group, $H \leq G$ and $S$ be a pronormal subgroup of $G$. If $N_G(S) \leq H$ then $H$ is pronormal in $G$.

DEFINITION. $H$ is a Hall subgroup of $G$ if $(|H|, |G:H|) = 1$.

Let $G$ be a group and $H \leq G$.

EXAMPLES. $H$ is a pronormal subgroup of $G$ in the following cases:

- $H$ is a normal subgroup of $G$;
- $H$ is a maximal subgroup of $G$;
- $H$ is a Sylow subgroup of $G$;
- $G$ is solvable and $H$ is a Hall subgroup of $G$;
- $S$ is a pronormal subgroup of $G$ and $N_G(S) \leq H$.

# PROBLEMS

QUESTION. What are pronormal subgroups of finite simple groups?

# CONJECTURE

DEFINITION. $H$ is a Hall subgroup of $G$ if $(|H|, |G : H|) = 1$.

THEOREM (Evgeny Vdovin and Danila Revin, 2012). All Hall subgroups are pronormal in finite simple groups.

The following conjecture was formulated by E. P. Vdovin and D. O. Revin.

CONJECTURE. Subgroups of odd indices are pronormal in finite simple groups.

EXAMPLE 1. Let $p$ doesn't divide $|G|$. In view of Glauberman's $Z^*$-Theorem in every finite simple group $G$ there exist two conjugate involutions $u$ and $v$ such that $uv = vu$. The subgroups $\langle u\rangle$ and $\langle v\rangle$ are conjugate in $G$ and every of them contain a (trivial) Sylow $p$-subgroup of $G$. But $\langle u\rangle$ and $\langle v\rangle$ are not pronormal since the subgroup $\langle u, v\rangle$ is abelian.

# SYLOW $p$-SUBGROUPS, WHERE $p > 2$

EXAMPLE 2. Let $p > 3$ and $G = A_{p+4}$. Consider conjugate subgroups

$$H_1 = \langle (1, ..., p)(p+1, p+2)(p+3, p+4) \rangle$$

and

$$H_2 = \langle (1, ..., p)(p+1, p+3)(p+2, p+4) \rangle$$

containing a Sulow $p$-subgroup of $G$. The subgroup $\langle H_1, H_2 \rangle \cong C_p \times V_4$ is abelian, so $H_1$ and $H_2$ are not pronormal in $G$.

# SYLOW $p$-SUBGROUPS, WHERE $p > 2$

EXAMPLE 3. Let $p = 3$, $G = M_{23}$, $H \cong PSL_3(4)\langle\tau\rangle$ is a subgroup of index 253 of $G$, $H_1 \cong A_6 \leq PSL_3(4)$, $H_2 = H_1^\tau$ and $H_1$ and $H_2$ are not conjugate in $PSL_3(4)$. $H_1$ and $H_2$ contain Sylow 3-subgroups of $G$ and $\langle H_1, H_2\rangle = PSL_3(4)$. So, $H_1$ and $H_2$ are not pronormal in $G$.

# PRONORMAL SUBGROUPS

PROPOSITION 1. Let $G$ be a group, $H \leq G$ and $S \leq H$ for some pronormal (possibly, Sylow) subgroup $S$ of $G$. Then the following conditions are equivalent:

(1) $H$ is pronormal in $G$;
(2) subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

REMARK. If $S$ is a Sylow subgroup of $G$ and $N_G(S) = S$ then $H$ is pronormal in $G$ for any subgroup $H \geq S$.

Recall, a non-trivial group $G$ is simple if it doesn't contain nontrivial proper normal subgroups.

Finite simple groups were classified. With respect to this classification, finite simple groups are:

- Alternating groups $A_n$ for $n \geq 5$;
- Classical groups $PSL_n(q) = L_n(q)$, $PSU_n(q) = U_n(q) = PSL_n^-(q) = L_n^-(q)$, $PSp_{2n}(q) = S_{2n}(q)$, $P\Omega_n(q) = O_n(q)$ ($n$ is odd), $P\Omega_n^+(q) = O_n^+(q)$ ($n$ is even), $P\Omega_n^-(q) = O_n^-(q)$ ($n$ is even);
- Exceptional groups of Lie type: $E_8(q)$, $E_7(q)$, $E_6(q)$, ${}^2E_6(q) = E_6^-(q)$, ${}^3D_{2n}(q)$, $F_4(q)$, ${}^2F_4(q)$, $G_2(q)$, ${}^2G_2(q)$ ($q$ is a power of 3), ${}^2B_2(q)$ ($q$ is a power of 2);
- 26 sporadic groups.

LEMMA (A. S. Kondrat'ev, 2005). Let $G$ be a finite nonabelian simple group and $S \in Syl_2(G)$. Then $N_G(S) = S$ excluding the following cases:

(1) $G \cong J_2,\ J_3,\ Suz$ or $HN$ and $|N_G(S) : S| = 3$;
(2) $G \cong {}^2G_2(3^{2n+1})$ or $J_1$ and $N_G(S) \cong 2^3.7.3 < \mathrm{Hol}(2^3)$;
(3) $G$ is a group of Lie type over field of characteristic 2 and $N_G(S)$ is a Borel subgroup of $G$;
(4) $G \cong L_2(q)$ where $3 < q \equiv \pm 3 \pmod 8$ and $N_G(S) \cong A_4$;
(5) $G \cong E_6^{\eta}(q)$ where $\eta = \pm$ and $q$ is odd and $|N_G(S) : S| = (q - \eta 1)_{2'}/(q - \eta 1, 3)_{2'} \neq 1$;
(6) $G \cong S_{2n}(q)$, where $n \geq 2$, $q \equiv \pm 3 \pmod 8$, $n = 2^{s_1} + \cdots + 2^{s_t}$ for $s_1 > \cdots > s_t \geq 0$ and $N_G(S)/S$ is the elementary abelian group of order $3^t$;
(7) $G \cong L_n^{\eta}(q)$, where $n \geq 3$, $\eta = \pm$, $q$ is odd, $n = 2^{s_1} + \cdots + 2^{s_t}$ for $s_1 > \cdots > s_t > 0$ and $N_G(S) \cong S \times C_1 \times \cdots \times C_{t-1}$, where $C_1, \ldots C_{t-2}, C_{t-1}$ are cyclic subgroup of orders $(q - \eta 1)_{2'}, \ldots, (q - \eta 1)_{2'}, (q - \eta 1)_{2'}/(q - \eta 1, n)_{2'}$ respectively.

THEOREM (A. Kondrat'ev, N.M., D. Revin, 2015). All subgroups of odd indices are pronormal in the following finite simple groups:

(1) $A_n$, where $n \geq 5$;
(2) sporadic groups;
(3) groups of Lie type over fields of characteristic 2;
(4) $L_{2^n}(q)$;
(5) $U_{2^n}(q)$;
(6) $S_{2n}(q)$, where $q \not\equiv \pm 3 \pmod 8$;
(7) $O_n^{\varepsilon}(q)$, where $\varepsilon \in \{+, -, \text{epmty symbol}\}$;
(8) exeptional groups of Lie type not isomorphic to $E_6(q)$ or ${}^2E_6(q)$.

PROBLEM. Are all subgroups of odd indices pronormal in the following finite simple groups:

(1) $L_n(q)$, where $n \neq 2^w$ and $q$ is odd;
(2) $U_n(q)$, where $n \neq 2^w$ and $q$ is odd;
(3) $S_{2n}(q)$, where $q \equiv \pm 3 \pmod 8$;
(4) exeptional groups of Lie type $E_6(q)$ and ${}^2E_6(q)$, where $q$ is odd?

# COUNTEREXAMPLE TO CONJECTURE

LEMMA (A. Kondrat'ev, N.M., D. Revin, 2015). Let $H$ and $V$ be subgroups of a group $G$ such that $V$ is an abelian normal subgroup of $G$ and $G = HV$. Then the following statements are equivalent:

(1) $H$ is pronormal in $G$;
(2) $U = N_U(H)[H,U]$ for any $H$-invariant subgroup $U \leq V$.

COROLLARY. Let $G = A \wr S_n = HV$, where $A$ is abelian, $H = S_n$ and $V = A^n$. Then the following statements are equivalent:

(1) $H$ is pronormal in $G$;
(2) $(|A|, n) = 1$.

# COUNTEREXAMPLE TO CONJECTURE

Let $q \equiv \pm 3 \pmod 8$ be a prime power and $n$ be a positive integer. It's well known, Sylow 2-subgroup $S$ of a group $T = Sp_2(q) = SL_2(q)$ is isomorphic to $Q_8$ and $N_T(S) \cong SL_2(3) = Q_8 : 3$. We have

$H = Q_8 \wr S_{3n} \leq X = SL_2(3) \wr S_{3n} \leq Y = Sp_2(q) \wr S_{3n} \leq G = Sp_{6n}(q)$.

PROPOSITION (N.M., 2008). If $L = Sp_n(q)$, where $n \geq 4$ and $P \cong Sp_m(q) \wr S_t \leq L$ then $|L : P|$ is odd if and only if $m = 2^w \geq 2$.

THEOREM (A. Kondrat'ev, N.M., D. Revin, 2015). The index $|G : H|$ is odd and $H$ isn't pronormal in $G$, so $H/Z(G)$ is a subgroup of odd index in $G/Z(G) \cong PSp_{6n}(q)$ which isn't pronormal.

THEOREM (A. Kondrat'ev, N.M., D. Revin, 2015). All subgroups of odd indices are pronormal in finite simple groups $PSp_{2^n}(q)$.

**Thank you for your attention!**