t-Walk-regular graphs, scheme graphs and 2-partially metric association schemes

Jack Koolen

This is based on joint work with M. Cámara, E.R. van Dam and J. Park, and with Zhi Qiao and Shao Fei Du

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Outline

Definitions

- Let $\Gamma = (V, E)$ be a graph.
- The distance $d(x, y)$ between two vertices x and y is the length of a shortest path connecting them.
- The maximum distance between two vertices in Γ is the diameter $D = D(\Gamma)$.
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- The valency of x is the number of vertices adjacent to it.
- A graph is regular with valency k if each vertex has k neighbors.
- The adjacency matrix A of Γ is the matrix whose rows and columns are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are adjacent and 0 otherwise.
- The eigenvalues of the graph Γ are the eigenvalues of A.

t-Walk-regular graphs

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- t-Walk-regular graphs are generalizations of distance-regular graphs. Many results on distance-regular graphs can be extended to the class of 2-walk-regular graphs, especially those results that uses Euclidean representations.

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Examples 1

There are many examples of m-walk-regular graphs that are not distance-regular.

- The bipartite double of the dodecahedron is 3-walk-regular but not 4-walk-regular. (Bipartite double: For every vertex x create two vertices x^+ and x^- and if $x \sim y$ then $x^\epsilon \sim y^\delta$ if $\epsilon\delta = -$.)

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- m-Arc transitive graphs are at least m-walk-regular. (m-Arc-transitive graphs have an automorphism group transitive on the m-arcs, i.e. $(m+1)$-tuples $(x_0, x_1, \ldots, x_m)$ such that $x_i \sim x_{i+1}$ and $x_{i-1} \neq x_{i+1}$)

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- But there are t-arc-transitive graphs which are $t+1$-walk-regular. For example all 1-arc-transitive cubic graphs are 2-walk-regular.
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- But there are t-arc-transitive graphs which are $t+1$-walk-regular. For example all 1-arc-transitive cubic graphs are 2-walk-regular.
- Any cubic graph is at most 5-arc-transitive (Tutte) and there are infinitely many connected non-isomorphic cubic 5-arc-transitive graphs.
- Any k-regular graph is at most 7-arc-transitive (Weiss) and there are infinitely many connected non-isomorphic 7-arc-transitive 4-regular graphs (Conder and Walker(1998))

Examples 2

Two generalizations of m-arc-transitive graphs:

- Partially m-distance-transitive graphs: Connected graph with diameter at least m such that for any quadruple of vertices x_1, x_2, y_1, y_2 with $d(x_1, x_2) = d(y_1, y_2) \leq m$ there is an automorphism τ such that $x_i^\tau = y_i$ $(i = 1, 2)$.

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- Praeger et al. (2010) also introduced the notion of m-geodetically-transitive graphs, i.e. the automorphism group is transitive on the $(m+1)$-tupels $(x_0, x_1, \ldots, x_m)$ with $x_i \sim x_{i+1}$ and $d(x_0, x_m) = m$.

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The same question for m-geodetically-transitive graphs.

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- We say Γ is t-partially distance-regular ($t \leq D$) (with partial intersection array $\iota = \{b_0, \ldots, b_t; c_1 = 1, c_2, \ldots, c_t\}$) if $\#\Gamma_{i-1}(y) \cap \Gamma_1(x) = c_i$ and $\#\Gamma_{i+1}(y) \cap \Gamma_1(x) = b_i$ for $d(x,y) = i \leq t$ with the understanding that $b_D = 0$.

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- If $t = D$, the graph is called distance-regular.

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- t-Walk-regularity is a global condition and t-partially distance-regularity is local condition.
- The last condition is much weaker then the first. Example: Take the folded n-cube $\tilde{Q}(n)$, i.e. you take the n-cube and you identify the antipodes. Take the cartesian product $K_2 \times \tilde{Q}(n)$. The resulting graph is about $n/2$-partially distance-regular but not even 1-walk-regular.

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- Assume that Γ has distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$.
- Then $\dim(\mathcal{A}) = d + 1$ and $\mathcal{A}$ has primitive idempotents E_i, $i = 0, 1, \ldots, d$ such that $AE_i = \theta_i E_i$.

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- Note a 0-walk-regular graph is regular say with valency $k(= b_0)$.

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Theorem

Let Γ be a connected 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Let x be a vertex of Γ and let $\Delta(x)$ has eigenvalues $a_1 = \eta_1 \geq \eta_2 \geq \ldots \geq \eta_k$.
Then $b^- := -1 - \frac{b_1}{1+\theta_1} \leq \eta_k \leq \eta_2 \leq b^+ := -1 - \frac{b_1}{1+\theta_d}$.

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- $i = 1$ or $i = d$.

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- $i = 1$ or $i = d$.
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- (Godsil) $k \leq (m_i + 2)(m_i - 1)/2$.

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C. Dalfó et al. (2011) showed the following result.

Proposition

Let s, d be positive integers. Let Γ be a connected s-walk-regular graph with diameter $D \geq s$ and with exactly $d+1$ distinct eigenvalues. Then the following hold:

- If $d \leq s+1$, then Γ is distance-regular;
- If $d \leq s+2$ and Γ is bipartite, then Γ is distance-regular.

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Later we will construct infinitely many bipartite 2-walk-regular graphs with 6 eigenvalues, which are not distance-regular. So this shows that we can not do better for $s = 2$ in the second item.

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- It is an open problem, whether there exist infinitely many 2-walk-regular graphs with exactly 5 distinct eigenvalues, which are not distance-regular.
- One way to construct them is to construct non-bipartite distance-regular graphs with diameter 3 and girth 6, and then take its line graph.

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- Take two distinct elements in X. If they are linearly dependent then there is no proper affine hyperplane they lie together in.
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- Take two distinct elements in X. If they are linearly dependent then there is no proper affine hyperplane they lie together in.
- If they are linearly independent then there are exactly q^{r-2} proper affine hyperplanes they lie together in.
- This shows:

- The design $\mathcal{D}(r,q) := (X, \mathcal{G}, \mathcal{B})$ is a group divisible design with the dual property with parameters $(q-1, \frac{q^r-1}{q-1}; q^{r-1}; 0, q^{r-2})$, or in other words a GDDDP$(q-1, \frac{q^r-1}{q-1}; q^{r-1}; 0, q^{r-2})$. (The dual property means that we can interchange the role of points and blocks to obtain a design with the same parameters).
- It is clear that the general linear group $\mathrm{GL}(r,q)$ acts as a group of automorphisms of $\mathcal{D}(r,q)$ such that its subgroup $Z := \{\alpha I_r \mid \alpha \in \mathrm{GF}^*(q)\}$ fixes the set G_x for all $x \in X$.

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- As we can consider $\mathrm{GF}(q^r)$ as a vector space of dimension r over $\mathrm{GF}(q)$, with basis $\{a^i \mid i = 0, 1, 2, \ldots, r-1\}$.
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- Now define the map $\tau_a \in$GL(r,q) by $\tau_a(x) = ax$ for $x \in \mathrm{GF}^*(q^r)$.
- Then τ_a generates a cyclic subgroup C of order $q^r - 1$ in GL(r,q).
- This shows that there is a cyclic group (the Singer group) of automorphisms that acts regularly on the points of the design. We will need this later.

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- $x \in X$ is adjacent to $B \in \mathcal{B}$ if x lies in B.
- This clearly gives a bipartite graph.

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- Du et al. classified the 2-arc-transitive dihedrants, but in their classification they did not have the graphs $\Gamma(r, q)$ with q even.
- $\mathcal{D}(r, q)$ can also be constructed using relative difference sets. That is how we found the examples of Bose.

Outline

Definitions 1

Let X be a finite set with n elements. A association scheme is a pair $(X, \mathcal{R})$ such that

(i) $\mathcal{R} = \{R_0, R_1, \cdots, R_d\}$ is a partition of $X \times X$,

(ii) $R_0 = \Delta := \{(x, x) | x \in X\}$,

(iii) for each i $(0 \leq i \leq d)$ there exists j such that $R_i = R_j^T$, i.e., if $(x, y) \in R_i$ then $(y, x) \in R_j$,

(iv) there are numbers p_{ij}^h (the intersection numbers of $(X, \mathcal{R})$) such that for any pair $(x, y) \in R_h$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^h.

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- The elements R_i are called the relations of $(X, \mathcal{R})$ and the number $d+1$ of relations is called the rank of $(X, \mathcal{R})$.
- If $R_i^T = R_i$, then we call the relation R_i symmetric.
- If all relations are symmetric, we call the scheme symmetric.

Definitions 2

- Let A_i be the relation matrix with respect to R_i such that the rows and the columns of A_i are indexed by the elements of X and the (x, y)-entry is 1 whenever $(x, y) \in R_i$ and 0 otherwise.

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Then the conditions (i)-(iv) are expressed by:

$(i)'$ $\sum_{i=0}^{d} A_i = J$, where J is the all-one matrix,

$(ii)'$ $A_0 = I$, where I is the identity matrix,

$(iii)'$ For all i there exists j such that $(A_i)^T = A_j$,

$(iv)'$ $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$.

- The Bose-Mesner Algebra $\mathcal{M}$ is the matrix algebra generated by the relation matrices (over $\mathbb{C}$).
- $\mathcal{M}$ has a basis of primitive idempotents called scheme idempotents if the scheme is symmetric.

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- An association scheme $(X, \mathcal{R})$ with rank $d+1$ is called t-partially metric (with respect to a symmetric relation R) if there exists an ordering of the relation matrices $A_0 = I, A_1, \cdots, A_d$ such that A_i is a polynomial of degree i in A for $i = 1, 2, \ldots, t$, where A is the relation matrix of R.

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- Note that $A_1 = A$ as the relation matrices are $(0, 1)$-matrices and that we assume that if $(X, \mathcal{R})$ is t-partially metric, then we always assume to have this ordering of the relation matrices $A_0 = I, A_1, \cdots, A_d$ such that A_i is a polynomial of degree i in A for $i = 1, 2, \ldots, t$.

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- A (symmetric) association scheme with rank $d+1$ is called metric if it is d-partially metric.

Definitions 4

We are going to construct graphs from association schemes.

- A graph Γ is called the scheme graph of $(X, \mathcal{R})$ (with respect to R) if the adjacency matrix A of Γ is equal to the relation matrix of R. In this case, we call the relation R the corresponding relation of Γ.
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- If relation R is the corresponding relation for a t-partially metric scheme, then the corresponding scheme graph is t-walk-regular.
- If the scheme is metric then the corresponding scheme graph is distance-regular.

Bipartite double

- The bipartite double of an association scheme $(X, R_0, R_1, \ldots, R_d)$ is the scheme $(X \times \{+, -\}, R_0^+, R_0^-, \ldots, R_d^+, R_d^-)$, where (x, ϵ) and (y, δ) are in relation $R_i^{\epsilon\delta}$ when x, y are in relation R_i.

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- If Γ is the scheme graph Γ of relation R_i in the original scheme, then the scheme graph of relation R_i^- is the bipartite double of Γ.

Outline

Examples 1

Most of the 2-partially metric association schemes come from groups. I will describe the scheme graphs of some examples.

- The t-arc-transitive graphs are scheme graphs of t-partially metric association schemes, but those schemes are usually not symmetric. These graphs have $c_2 = 1$ if $t \geq 2$.

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- The bipartite double of the dodecahedron is the scheme graph of two different symmetric association schemes, namely the bipartite double scheme BD of the metric scheme of the dodecahedron and a fusion scheme of BD. The scheme BD is 2-partially metric, whereas the latter scheme is 3-partially metric.

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- The symmetric bilinear forms graphs $\mathrm{SBF}(n, q)$ have as vertices the $n \times n$ symmetric matrices over a finite field $\mathrm{GF}(q)$ (where q is a prime power) and two matrices are adjacent if their difference has rank 1. These graphs have $c_2 \geq 2$ and are locally the disjoint union of cliques. For $n \geq 4$ they are 2-distance-transitive but not distance-regular.

Examples 2

- De Caen et al. found an infinite family of triangle-free distance-regular antipodal graphs of diameter 3. If you take the bipartite double of these graphs you obtain 2-walk-regular graphs with $c_2 = 2$ and $a_1 = 0$. They are also the scheme graphs of the bipartite double scheme of the underlying metric scheme, and this scheme is 2-partially metric and symmetric.

Examples from codes

- Let C be a binary linear code, say of length n, i.e. a subspace of the n-dim space $\mathrm{GF}(2)^n$.
- Let $\Gamma(C)$ be the coset graph of C, i.e.the vertices are the cosets $x + C$ of C and two cosets are adjacent if there is an edge between them in the Hamming graph.

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- But usually the coset graph $\Gamma(C)$ is not 2-walk-regular.
- If the automorphism group of the code C acts 2-transitive on the positions and the minimum weight is at 4, then $\Gamma(C)$ is partially 2-distance-transitive.

Examples from codes 2

- Let C be the truncated code of the even sub code of the Golay code. Then $\Gamma(C)$ is distance-transitive.

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- The bipartite double of $\Gamma(C)$ is 3-distance-transitive, and is the coset graph of the even sub code of C.
- There are some more examples which can be constructed from certain sub codes of the Golay, but those that are 3-distance-transitive are also distance-transitive.
- Let C be the simplex t-dimensional code over the binary field, i.e. the dual code of a Hamming code of length $2^t - 1$. Then the coset graph $\Gamma(C)$ is 2-distance-transitive but not 3-walk-regular.
- There are many more examples of coset graphs that are 2-distance-transitive. We are still working in this.

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- Let $\mathcal{D}$ be a 2-design for which the automorphism group of the design acts 2-transitive on the points.
- Let C be the linear code generated by the support of the blocks. Note it makes a difference here whether you look at the design or at its complementary design, i.e. the blocks are the complements of the blocks of the original design.

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- Let $\mathcal{D}$ be a 2-design for which the automorphism group of the design acts 2-transitive on the points.
- Let C be the linear code generated by the support of the blocks. Note it makes a difference here whether you look at the design or at its complementary design, i.e. the blocks are the complements of the blocks of the original design.
- For example take as your design the projective plane of order a power of 2. (For odd order you obtain the trivial code $\mathrm{GF}(2)^n$.) The dimension of this code has been determined long ago by many people. We can show the coset graph is 2-distance-transitive. We are still trying to determine whether they are 3-walk-regular.

Outline

2-Walk-regular graphs with multiplicity 3

Theorem 1 [2013,CDKP]

Let Γ be a 2-walk-regular graph, different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$ with multiplicity 3.

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Cubic 2-walk-regular graphs?????

An family of cubic 2-walk-regular graphs with multiplicity 3

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But if you assume that the graph comes from a symmetric association scheme with a connecting relation with valency 3, then we can classify them.

Symmetric association schemes with multiplicity 3

Our Result:

Theorem 2

Let $(X, \mathcal{R})$ be a 2-partially metric association scheme with corresponding relation R, corresponding valency $k \geq 3$ and rank $d+1 \geq 3$. Let $E_0, E_1 \cdots, E_d$ be the minimal scheme idempotents with corresponding eigenvalues $\theta_0 = k, \theta_1 \ldots, \theta_d$ and multiplicities $m_0 = 1, m_1, \ldots, m_d$ respectively. Let Γ be the scheme graph of $(X, \mathcal{R})$. If there exists an integer i $(1 \leq i \leq d)$ such that $m_i = 3$, then one of the following holds:
(i) Γ is the cube,
(ii) Γ is the Möbius-Kantor graph (a 2-cover of the cube),
(iii) Γ is the Nauru graph (a 3-cover of the cube),
(iv) Γ is the dodecahedron,
(v) Γ is the bipartite double of the dodecahedron,
(vi) Γ is the icosahedron,
(vii) Γ is the octahedron,
($viii$) Γ is a regular complete 4-partite graph.
Moreover, the association scheme is uniquely determined by Γ.

Remarks

- The bipartite double of the dodecahedron has no eigenvalue with multiplicity 3. As I remarked earlier, there are two symmetric 2-partially metric association schemes with this graph as its scheme graphs, namely the bipartite double scheme of the dodecahedron and a fusion scheme of this scheme.

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- The first scheme has four minimal idempotents with multiplicity 3, two of them have corresponding eigenvalue $\sqrt{5}$ and two of them have corresponding eigenvalue $-\sqrt{5}$.

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- If the valency k equals three, then we do not need to assume that the scheme is 2-partially metric, as that is implied by a result of N. Yamazaki (1998).
- The theorem is not true for symmetric association schemes (which are not 2-partially metric), as the t-coclique extensions of the dodecahedron show. (In this case the smallest non-trivial valency is equal to $3t$)

Related work

Ei. Bannai and Et. Bannai (2006) showed the following result:

Theorem 3

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N. Yamazaki (1998) studied the symmetric association schemes with a relation with valency three. He showed:

Theorem 4

Let $(X, R_0, \ldots, R_d)$ be a symmetric association scheme with a connecting relation R of valency 3. Then the association scheme is metric with respect to R (and its corresponding scheme graph is distance-regular), or the corresponding scheme graph is bipartite.

Outline

Problems

We conclude this talk with some open problems.

- Are there only finitely many symmetric association schemes with a connecting relation with valency 3?
- For fixed $k \geq 3$, is 2-partially metric enough to show that there only finitely many symmetric association schemes with a connecting relation with valency k?

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- Are there only finitely many symmetric association schemes with a connecting relation with valency 3?
- For fixed $k \geq 3$, is 2-partially metric enough to show that there only finitely many symmetric association schemes with a connecting relation with valency k?
- Find more examples of 3-partially metric symmetric schemes, which are not metric. On this moment we only know of about 3 or 4 examples.
- Find 2-walk-regular graphs which are locally connected. On this moment we do not have any example which is not distance-regular.

Thank you for your attention.